Let $v = (x^2+1)^{1/2}$. Then

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2x^3-5}{v}\right) = \left(6x^2v - (2x^3-5)\frac{dv}{dx}\right)\Big/ v^2$$

by the quotient rule. By the composite function rule and the extended power rule,

$$\frac{dv}{dx} = \frac{1}{2}(x^2+1)^{-1/2} \times 2x = \frac{x}{v}.$$

Therefore

$$\frac{dy}{dx} = (6x^2v^2 - (2x^3-5)x)/v^3$$

$$= (6x(x^2+1) - 2x^3+5)\frac{x}{v^3}$$

$$= \frac{(4x^3+6x+5)x}{(x^2+1)^{3/2}}.$$

Two final points about the rules

1. There is often more than one way of differentiating a particular function: thus in Example 1 we could use the quotient rule instead of the composite function rule.
2. It often saves time to **simplify before differentiating**; in particular, this avoids unnecessary use of the product and quotient rules. For example, the function $y = (5x^2+2x)x^2$ could be differentiated using the product rule but it is much easier to write $y = 5x^4+2x^3$ and then differentiate: $dy/dx = 20x^3+6x^2$. Similarly, the function $y = (5x^2+2x)/x^2$ should not be differentiated by the quotient rule; instead, we write $y = 5+2x^{-1}$, and the derivative is easily seen to be $-2/x^2$.

Exercises

7.2.1 The functions f and g are defined as follows:

$$f(x) = x^3+1, \quad g(x) = x^4-2.$$

Find expressions for $f(g(x))$, $g(f(x))$ and their derivatives.

7.2.2 Differentiate:

(a) $(3x-7)^{10}$ (b) $(x^3+1)^5$ (c) $(4x+9)^{1/2}$ (d) $(x^6-1)^{2/3}$ (e) $(x^{1/4}+5)^6$
(f) $(x^4-3x^2+5x+1)^{1/4}$ (g) $1/(x^2-1)^7$ (h) $8/(\sqrt{x}+2)^5$

7.2.3 Differentiate using the rules of this and the previous section:

(a) $(x^2-1)(x^3+1)^5$, (b) $(x^{1/3}-2)/(x^5-2)^3$.

7.2.4 Suppose the demand function for

$$q = 3p^{-1/4},$$

where p denotes price and q quantity supplied. Find the elasticity of demand. [The same method gives the elasticity of demand for any demand function of the form $q = ap^{-b}$, where a and b are positive real numbers. The result of part (ii) of the final part of Problem 6–4 is true in this more general case.]

7.2.5 Suppose that a firm's output Q is related to labour input L by the production function

$$Q = L^{2/5}.$$

Suppose further that L is given by the linear function

$$L = 4+3t.$$

Find dQ/dt.

7.2.6 Using the composite function rule, show that

$$\frac{d}{dx}\left(\frac{1}{v}\right) = -\frac{1}{v^2}\frac{dv}{dx}.$$

Using this result and the product rule, derive the quotient rule.

7.2.7 In Example 2 of Section 7.1, we used the quotient rule to show that

$$\text{if } y = \frac{x^2}{x^2-x+1} \text{ then } \frac{dy}{dx} = \frac{x(2-x)}{(x^2-x+1)^2}.$$

Obtain the same result by writing $y = u^{-1}$, where $u = 1-x^{-1}+x^{-2}$, and using the composite function rule.

7.3 Monotonic functions

Some functions have graphs that slope upward: $y = x$ is an obvious example. Others have negative slope wherever they are defined, for example demand functions in economics. In this section we explain the main properties of these two kinds of function.

We say that the function f is **strictly increasing** if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$. Similarly, f is said to be **strictly decreasing** if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$. In either case f is said to be a **monotonic function.**

The **domain** of a monotonic function f (in other words, the set of all x for which $f(x)$ is defined) may be all of $\mathbb{R}$ or an interval of real numbers. The four panels of Figure 7.1 show the graphs of four monotonic functions, two strictly increasing and two strictly decreasing. The domain of the functions graphed in panels (A) and (B)