Chapter 31

INTRODUCTION TO ANALYSIS

The branch of mathematics known as analysis is concerned with the logical foundations of calculus. In the last two chapters of this book, we give a very short introduction to real analysis, the part of the field based on real rather than complex numbers.

Real analysis is much used in modern economic theory, both macro and micro. This development is motivated by a concern with whether particular mathematical problems arising in economics have solutions, and its central feature has been an emphasis on mathematical rigour. The issue of existence of solutions is one we have largely avoided up to now, and we shall discuss it in the second half of the next chapter. Such a discussion requires preparation, and this chapter provides the tools and language. We begin by explaining what rigour is and then proceed to a more thorough treatment of the real numbers, limits and continuity than we gave in Chapters 3 and 5.

Before studying this chapter you may find it helpful to refresh your memory of Section 5.4.
When you have studied this chapter you will understand:

- the key properties of the real number system;
- the concept of boundedness, in the context of $\mathbb{R}$;
- the key facts about convergence of sequences of real numbers, and their proofs;
- the properties of continuous functions of a real variable, and their proofs.

31.1 Rigour

Mathematically rigorous arguments have two ingredients. First, assumptions must be spelt out carefully. In particular, this requires that definitions be precise.

In the preceding 30 chapters we have been reasonably careful about stating assumptions; the main exceptions have occurred in calculus, where we often assumed implicitly and without justification that functions could be differentiated or integrated.

Much of mathematical analysis is concerned with precisely these issues of differentiability and integrability, but we shall not discuss those matters further.[1] With respect to definitions we have also been quite precise, the main exception being our failure to give a serious definition of a real number: the nearest we got to one was our rather vague comment in Section 3.2 about filling in the gaps between the rational numbers. In this chapter we shall again stop short of a complete description of $\mathbb{R}$; we shall however be sufficiently explicit about what 'filling in' means to provide a foundation for rigorous analysis.

The second ingredient of rigour is logical argument: given two statements, we need to be able to say whether or not one follows from the other. This frequently involves a careful distinction between necessary and sufficient conditions. The distinction has been mentioned earlier at various places: see for instance our discussion of conditions for maxima and minima in Section 8.3. We now discuss relationships between statements in a more systematic way.

If statement Q follows from statement P we write $P \Rightarrow Q$. The logical symbol $\Rightarrow$ means 'implies', so we can also read $P \Rightarrow Q$ as 'P implies Q'. Other ways of conveying the information that $P \Rightarrow Q$ include the following:

- if P is true then Q is true;
- P is a sufficient condition for Q;
- Q is a necessary condition for P;
- Q is implied by P.

The symbol $\Leftarrow$ means 'is implied by': thus $Q \Leftarrow P$ means the same thing as $P \Rightarrow Q$.

If $P \Rightarrow Q$ and $Q \Rightarrow P$, we say that P is **equivalent** to Q and write $P \Leftrightarrow Q$. Other ways of expressing the relation $P \Leftrightarrow Q$ include:

- P is true if and only if Q is true;
- P is a necessary and sufficient condition for Q;
- $Q \Leftrightarrow P$.

As an example of the logical symbols in action, consider the following statements.

P Mary lives in London

Q Mary lives in England

R Mary lives in the capital city of England

S Mary lives in a capital city

Here $P \Rightarrow Q$, $P \Leftrightarrow R$, $P \Rightarrow S$, $Q \Leftarrow R$ and $R \Rightarrow S$. No pair of the four statements other than P and R is connected by the relation $\Leftrightarrow$. The statements Q and S are not connected by $\Rightarrow$ or $\Leftarrow$.

[1]They are treated at length in the textbooks by Binmore and Rudin cited in the Notes on Further Reading at the end of this book.

Mathematical theorems usually take the form $P \Rightarrow Q$ or $P \Leftrightarrow Q$. For a theorem of the form $P \Rightarrow Q$, statement P is known as the assumptions or premises and statement Q is known as the conclusion. The proof of such a theorem can be carried out by assuming P, making a series of deductions and finally showing that Q must hold. Alternatively, we can assume that Q is false and show P must be false: this is known as **proof by contraposition**.[2] The proof of a theorem of the form $P \Leftrightarrow Q$ typically involves two distinct steps: proof that $P \Rightarrow Q$, and proof of the **converse** or **reverse implication** $P \Leftarrow Q$.

To show that P does *not* imply Q, all we have to do is to produce one situation in which P holds but Q does not. This is called **giving a counterexample.**

To illustrate these principles, we consider the following two propositions:

(a) if x and y are real numbers, then $(x+y)^2 = x^2 + y^2 + 2xy$;

(b) if x and y are real numbers, then $(x+y)^2 = x^2 + y^2$.

As you must know by now, (a) is true and (b) is false. (a) is proved by multiplying out the brackets in the expression $(x+y)(x+y)$. The important point is that this must be done algebraically, with x and y interpreted as arbitrary real numbers; verifying that the conclusion holds for particular numerical values of x and y does not constitute a proof. The reason for this is that (a) declares something to be true *for all* real numbers x and y. By contrast, to prove that (b) is false it suffices to show that the conclusion is false *for some* real numbers x and y: $x = y = 1$ will do.

The phrases 'for all' and 'for some' are so important in mathematics that there is a standard notation for them: $\forall$ means 'for all' and $\exists$ means 'there exists'. For example, (a) above may be written

$$(x+y)^2 = x^2 + y^2 + 2xy \ \ \forall x, y \in \mathbb{R},$$

while the fact that (b) is false may be expressed as follows:

$$\exists x, y \in \mathbb{R} \text{ such that } (x+y)^2 \neq x^2 + y^2.$$

The symbols $\forall$ and $\exists$ are known as **quantifiers** because they indicate the quantity of cases for which a given statement is true.

Mathematical induction

Let $\mathbb{N}$ denote as usual the set of all natural numbers. Suppose we are trying to prove a theorem of the form 'for all $n \in \mathbb{N}$, P_n', where P_n is a statement concerning n. A common way of proceeding is in two steps: (a) prove P_1; (b) prove that $P_n \Rightarrow P_{n+1} \ \forall n \geq 1$. This is known as **proof by induction**: (a) is known as the initial step, (b) as the induction step.

[2]Contraposition is often used in cases where Q takes the form '*not* R': to prove that $P \Rightarrow$ *not* R, one shows that $R \Rightarrow$ *not* P. A classic example is the proof of the irrationality of $\sqrt{2}$ that we gave in Section 3.3. Here P is the statement '$k = 2$' and R is the statement that k is the square of a rational number.

To illustrate this procedure, we give an alternative proof of the usual formula (5.2) for the sum of the first n natural numbers. For each natural number n, let $i_n = 1+\ldots+n$ and let P_n be the statement that $i_n = \frac{1}{2}n(n+1)$; we wish to prove that P_n is true for all such n. P_1 is true because $\frac{1}{2}\times 1\times 2 = 1$. It remains to perform the induction step: if P_n is true, then

$$i_{n+1} = i_n + (n+1) = \tfrac{1}{2}n(n+1) + (n+1) = (n+1)(\tfrac{1}{2}n+1) = \tfrac{1}{2}(n+1)(n+2),$$

so P_{n+1} holds as required.

Mappings, lists and sequences

The notions of set and subset were introduced at the beginning in Chapter 3; recall in particular the definition of the empty set $\varnothing$, and the distinction between finite and infinite sets.

Also in Chapter 3, we defined a **mapping** from a set A to a set B to be a rule that transforms each element of A into an element of B. The **domain** of f is the set A; as we said in Chapter 3, the **range** of f is the set $\{\, f(x) : x \in A\}$, which is a subset of B. A common abbreviation for 'the mapping f from A to B' is 'the mapping $f\colon A \to B$'. In this chapter and the next, we shall refer to a mapping f only as f, and not as $f(x)$ or '$y = f(x)$': the looser conventions of earlier chapters are useful for many purposes, but not for analysis. A **function** is a mapping whose range is a set of numbers: as we stated in Section 3.4, this convention for distinguishing between functions and mappings is common but not universal.

Ordered pairs and lists of real numbers were also introduced in Chapter 3. The general notion of an ordered pair does not require that it consist of two real numbers, or even two members of the same set. Given two sets X and Y, the **Cartesian product** $X\times Y$ is defined to be the set of all ordered pairs (x, y), where $x \in X$ and $y \in Y$. For example, if C is the set of all names of European capital cities, then $(\text{London}, 4) \in C\times\mathbb{N}$. Notice that $(7, \text{Rome})$ is not a member of $C\times\mathbb{N}$, since we are talking about *ordered* pairs; $(7, \text{Rome})$ is in fact a member of the set $\mathbb{N}\times C$. Similarly, if $X_1, X_2, \ldots, X_n$ are sets, then the Cartesian product $X_1\times X_2\times\ldots\times X_n$ is the set of all ordered lists of the form $(x_1, x_2, \ldots, x_n)$, where $x_i \in X_i$ for $i = 1, 2, \ldots, n$. Notice a connection between lists and mappings: given a set X, a member of the n-fold Cartesian product $X\times X\times\ldots\times X$ may be regarded as a mapping from the set $\{1, 2, \ldots, n\}$ to X.

A **sequence** of members of a set X is a mapping from $\mathbb{N}$ to X; thus 'the sequence $\{u_n\}$' refers to a mapping $f\colon \mathbb{N} \to X$ such that $f(n) = u_n\ \forall n \in \mathbb{N}$. This is the definition of a sequence given in Section 5.1, except that we no longer restrict attention to the case where $X = \mathbb{R}$.

The sequence $\{u_n\}$ is not the same object as the set $\{\, u_n : n \in \mathbb{N}\}$. One reason why the difference matters is as follows: all sequences in this chapter are 'infinite' in the sense that u_n is defined for *every* natural number n, but the set $\{\, u_n : n \in \mathbb{N}\}$ may be finite. This happens, for example, if u_n is defined to be 1 for all odd n and 0 for all even n.

Exercises

31.1.1 As in the text, let $\mathbb{Q}$ be the set of all rational numbers. If $x \in \mathbb{Q}$, which of the following statements imply which of the others?

P $x \geq 0$.

Q $\exists w \in \mathbb{Q}$ such that $x = w^2$.

R $nx > -1 \ \forall n \in \mathbb{N}$.

S $\exists n \in \mathbb{N}$ such that $nx > -1$.

31.1.2 Prove by induction that

$$(1+a)^n \geq 1 + na + \tfrac{1}{2}n(n-1)a^2$$

whenever $a > 0$ and $n \in \mathbb{N}$. [This result can also be derived from the binomial theorem, which can itself be proved by induction.]

31.1.3 Let I be an interval of real numbers, and let the function $f\colon I \to \mathbb{R}$ be concave. Prove by induction that if $x_1, x_2, \ldots, x_n$ are members of I and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are positive numbers such that $\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1$, then

$$f(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n) \geq \alpha_1 f(x_1) + \alpha_2 f(x_2) + \ldots + \alpha_n f(x_n).$$

31.1.4 Let $I = \{x \in \mathbb{R} : 0 \leq x \leq 2\}$, $J = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$. Illustrate in a single diagram the sets $I \times J$ and $J \times I$.

31.2 More on the real number system

A rigorous treatment of limits and continuity requires a more detailed description of the real number system than that given in Section 3.2. We now give such a description, though — as we remarked earlier — we shall not give a fully formal definition of $\mathbb{R}$.

The real number system $\mathbb{R}$ is characterised by three properties, of which the first two are the **laws of arithmetic** and the **laws of order**. The former are the familiar rules of addition, subtraction, multiplication and division and the special properties of the numbers 0 and 1.

The laws of order for $\mathbb{R}$ are the rules for working with inequalities given in Section 2.1, the fact that $x^2 \geq 0$ for any real numb