Problems on Chapter 9

9–1. As a generalisation of Example 4 of Section 9.2, consider the function

$$y = \exp(-ax^2),$$

where a is a positive constant.

(i) Find dy/dx. Without calculating the second derivative, show that the unique global maximum is at $x = 0$.

(ii) Find d^2y/dx^2. Hence find the points of inflexion and the ranges of values for which the function is convex and concave. Sketch the graph.

9–2. (i) A sum of money is invested at a constant (instantaneous) rate of interest r per year, compounded continuously. Let $R = 100r$, so that the instantaneous rate of interest, treated as a percentage, is R% per annum. Suppose the investment is worth double at the end of T years.[6] Show that

$$T \approx \frac{69}{R}.$$

(ii) The approximation in (i) is known in finance as the **rule of 69**. It is often convenient to use an alternative approximation for the doubling time in which the denominator is the APR rather than the instantaneous flat rate; the numerator should then be somewhat greater than 69, and increase with the APR. Show that the 'rule of 70' provides a reasonable approximation if the APR is below 5% per annum, and the 'rule of 72' if it is 5–10%. [Hint: use part (i), and your answer to Exercise 9.2.1.]

9–3. (i) By expressing the limit as the value of a derivative at $x = 0$, show that

$$\lim_{x\to 0} \frac{e^{ax} - 1}{x} = a.$$

Hence show that

$$\lim_{a\to 0} \frac{e^{ax} - 1}{a} = x.$$

(ii) For each real number $a \neq 0$, the function f_a is defined as follows:

$$f_a(x) = \frac{e^{ax} - 1}{a}.$$

Also let $f_0(x) = x$: by the second result of (i),

$$\lim_{a\to 0} f_a(x) = f_0(x) \quad \text{for all } x.$$

Find $f_a'(x)$, $f_a''(x)$, $f_a(0)$ and $f_a'(0)$. Use these results to sketch the curves $y = f_a(x)$ in the same xy–plane for $a = 0, \pm 1$ and ± 5.

[6] Since time is continuous, T does not have to be an integer.