(iii) For each real number $b \neq 0$, the function g_b is defined as follows:

$$g_b(x) = \frac{x^b - 1}{b} \quad (x > 0).$$

Also let $g_0(x) = \ln x$. Use the second result of (i) to show that

$$\lim_{b \to 0} g_b(x) = g_0(x) \quad \text{for all } x > 0.$$

Find $g_b'(x)$, $g_b''(x)$, $g_b(1)$ and $g_b'(1)$. Use these results to sketch the curves $y = g_b(x)$ in the same xy-plane for $b = 0, \frac{1}{2}, \pm 1$ and ± 2.

[The functions of part (ii) are known in financial economics as 'constant absolute risk aversion' (CARA) utility functions: the coefficient of absolute risk aversion is $-a$. The functions of part (iii) are known in statistics as 'Box–Cox transformations' and in financial economics as 'constant relative risk aversion' (CRRA) utility functions: the coefficient of relative risk aversion is $1 - b$.]

9–4. (i) As in part (i) of Problem 5–4, a timber owner plants a forest at time 0. The volume of timber in the forest at time $t \geq 0$ is given by the function $f(t)$, The price per unit volume of wood harvested is p and the interest rate is r. Both p and r are assumed to be constant, p is net of harvesting costs and planting costs are ignored.

Suppose now that time is measured continuously and discounting is continuous, with r being the instantaneous rate of interest. If the forest is cut down and the timber sold at time T, what is the value of the forest at time 0? Show that this value, considered as a function of T, has a critical point where T satisfies

$$f'(T)/f(T) = r. \tag{9.10}$$

[Equation (9.10) is called the **Fisher rule**, after the famous American economist Irving Fisher (1867–1947).]

(ii) Assume everything is as in (i), except that at time T, $2T$, $3T$, ... the timber owner cuts down the forest, sells the timber and replants it with similar trees.[7] Express the value of the forest at time 0 as a function of T, and show that it has a critical point where

$$f'(T)/f(T) = re^{rT} / (e^{rT} - 1) . \tag{9.10'}$$

[This equation is called the **Faustmann rule**, after the nineteenth-century German landowner and forester Martin Faustmann.]

(iii) Show that the right-hand side of (9.10′) approaches $1/T$ as $r \to 0$.
[This means that, when the interest rate is very low, the Faustmann rule approximates the 'forester's rule' of Problem 8–2.]

(iv) Assuming $f(t)$ has the shape specified in Problem 8–2, discuss the circumstances in which the stationary points of parts (i) and (ii) are global maxima.

[7] T is called the **rotation period**, and does not have to be an integer.