Exercise 5.2 & 5.3

1. **(a)** This is an existence statement. It is enough to find such an integer, e.g. $20 = 4 + 6 + 10$

(b) Direct proof: Let an even integer be $2m$ where m is any integer. m can be written as the sum of any 3 integers, p, q and r. Thus $2m = 2(p+q+r) = 2p+2q+2r$, which are even integers.

(c) Not true. Let $(2n + 1)$ and $(2k + 1)$ be any two odd integers. Then, $(2n + 1) + (2k + 1) = 2(n + k + 1)$ which is even. Thus, the sum cannot be odd.

(d) Let $m = 2n + 1$ be any odd integer.
$2n$ is even, thus it can be the sum of three even integers, so,
$m = 2p+2q+2r+1 = (2p+1)+(2q-1)+(2r+1)$, which are three odd integers.

(e) Let $n, n+1,$ and $n+2$ be three consecutive integers, then,
$n+(n+1)+(n+2) = 3n+3 = 3(n+1)$, which is a multiple of 3.

(f) Not true. A counter example is enough: $n+(n+1)+(n+2)+(n+3) = 4n+6$.
If $n = 1$, this sum is 10, which is not divisible by 4.

(g) True. Every three consecutive numbers should have at least one even number and every three consecutive numbers should have one multiple of 3 (multiples of 3 are periodic with period 3). Thus, the product should be a multiple of 6.

(h) Not true since $a = -b$ is another possibility.

2. **(a)** We need to find the perfect square less than or equal to 1871. This is 1849 or 43^2. That is, he turned 43 in year 1849, so he was born in $1849 - 43 = 1806$.

(b) Again, we need to find the perfect square less than or equal to 2018. The closest perfect square less than 2018 is $1936 = 44^2$. This means that your friend turned 44 in year 1936, which means he/she will be 126 in year 2018. This is not likely given our life expectancy! (The greatest fully authenticated age to which any human has ever lived is 122 years 164 days by Frenchwoman Jeanne Louise Calment.)

3. If $n \geq 2$, then $n!$ contains at least one even factor. Thus, for example we factor 2 out of both terms $n(n-1)\times(n-2)\times\cdots\times 2\times 1 + 2 = 2(n(n-1)\times(n-2)\times\cdots\times 1 + 1) = 2Q$, which is even.

4. **(a)** Needs to be proven in both directions.
If $5n + 3$ even, then if you add any odd number to it, the resulting number will be odd. So, adding the odd integer $2n - 5$ to $5n + 3$ will result in an odd integer. That is, $5n + 3 + 2n - 5 = 7n - 2$ is odd. Similarly, if $7n - 2$ is odd, adding the odd integer $5 - 2n$ will result in an even number. That is, $7n - 2 + 5 - 2n = 5n + 3$ is even.

(b) $7n - 2$ will be even. If $5n + 3$ is odd, then $5n$ must be even $\Rightarrow n$ is even $\Rightarrow 7n - 2$ is even.

5. $m^2 + n^2$ even $\Rightarrow m^2$ and n^2 are both even or both odd $\Rightarrow m$ and n have the same parity because if, for example, m^2 is even, m must be even as the product of two odd numbers cannot be even. Similarly, if m^2 is odd, then m must be odd.

6. By contradiction:
Assume that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$. Square both sides and simplify, $0 = xy$. Thus, one of x or y must be zero, which contradicts the fact that both are positive.

7. If $x = 0$, or $y = 0$, then $xy = 0$ is obvious.
In the opposite direction: If $xy = 0$. Assume that x and y are both different from zero.
But there are no nonzero real numbers that can have a product of zero, thus a contradiction.

8. Let O be the set of odd numbers.
(a) $\exists\, x \in O$ such that $x = k^2$ where, $k \in \mathbb{Z}$
(b) $\forall\, x \in O$, $x \neq k^2$ where, $k \in \mathbb{Z}$
(c) True. Since it is an existence statement, $x = 81 = 9^2$

9. **(a)** Statement: $\forall\, x \in \mathbb{Z}^+$, $13 \mid x$. Negation: $\exists\, x \in \mathbb{Z}^+$, $13 \nmid x$
(b) False, a counter example: $x = 10$, $13 \nmid 10$

10. By contradiction: Assume $a^2(b^2 - 2b)$ is odd, but at least a or b is even.
Say a is even, then, $a = 2k$ and $a^2(b^2 - 2b) = 4k^2(b^2 - 2b)$ is even. Contradiction.

11. Contrapositive: $5 \mid m$ and $5 \mid n \Rightarrow m = 5r$ and $n = 5s \Rightarrow mn = 25rs \Rightarrow 25 \mid mn$

12. Prove by cases: $m + n$ is even $\Rightarrow m$ and n must both be odd or both be even.
1. If both are odd: m^2 is odd and n^2 is odd $\Rightarrow m^2 + n^2$ is even.
2. If both are even: m^2 is even and n^2 is even $\Rightarrow m^2 + n^2$ is even.

Or, direct: $m + n$ is even $\Rightarrow m + n = 2k \Rightarrow m^2 + n^2 = (m + n)^2 - 2mn = 4k^2 - 2mn = 2N$

13. If n is even, then $n = 2k \Rightarrow n^2 + 2n + 9 = 4(k^2 + k) + 9$, which is odd.
If $n^2 + 2n + 9$ is odd, then $n^2 + 2n$ must be even. But $2n$ is even, so n^2 is even, and so, n must be even. (Or, by using the contrapositive method.)

14. Let $a = 2k + 1$ and simplify. $a^2 + 3a + 5 = 2(2k^2 + 5k + 4) + 1 = 2N + 1$, which is odd.