satisfying these two inequalities and the budget constraint is called the **budget set**. The **budget line** is the straight line with equation

$$p_1x_1 + p_2x_2 = m.$$

Points on the budget line whose coordinates are both non-negative represent consumption choices such that Ian spends all his income.

To sketch the budget set in the positive quadrant of the x_1x_2–plane, we begin by drawing the part of budget line in that quadrant. The budget line meets the horizontal axis where $x_1 = m/p_1$ and the vertical axis where $x_2 = m/p_2$. By using the origin as test point in the usual way, we see that the set of all points satisfying the budget constraint consists of the budget line and those points on the same side of it as the origin. The budget set is drawn shaded in Figure 2.5.

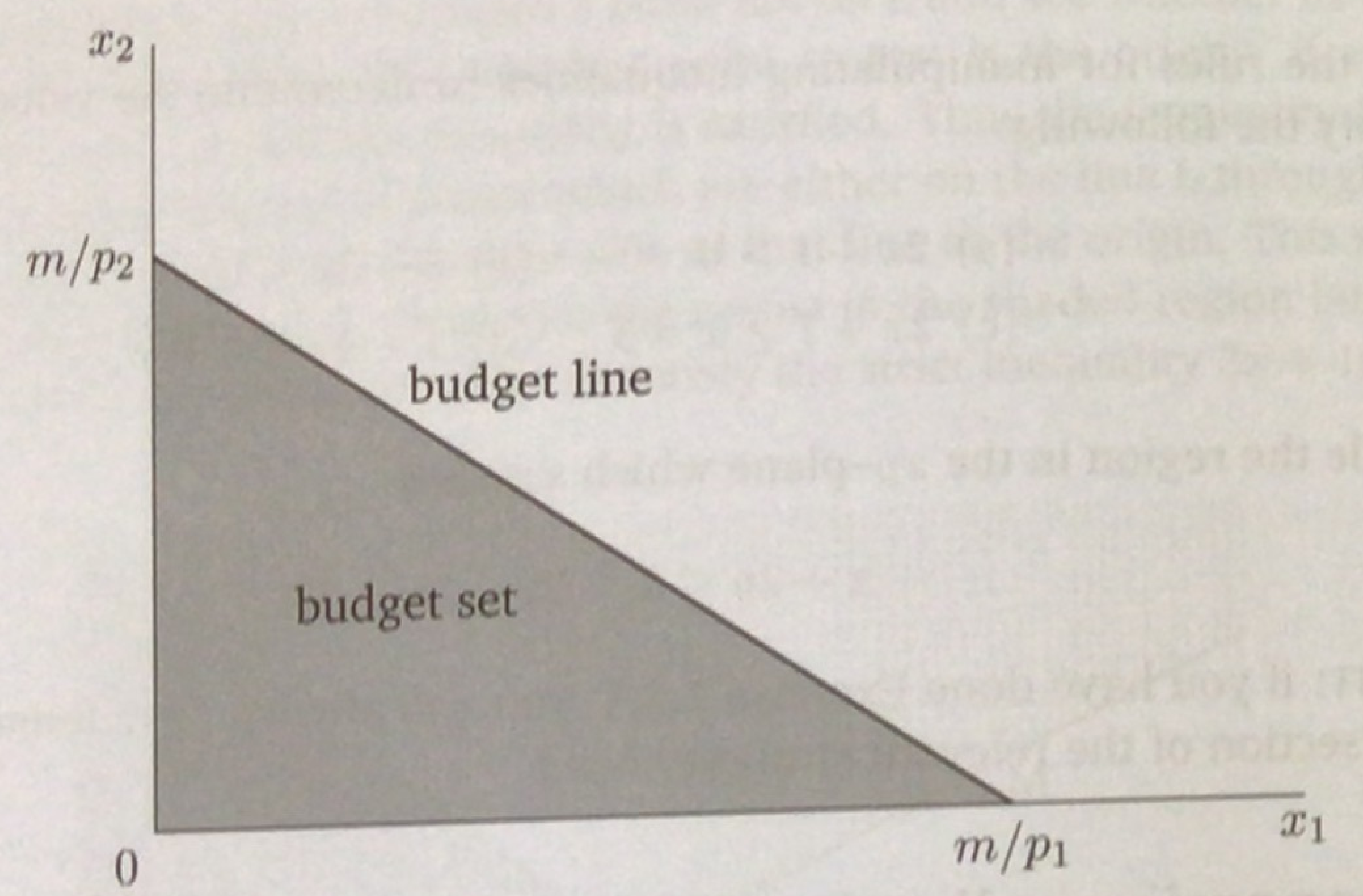

Figure 2.5: The budget line and the budget set

Sketching production possibilities

Suppose a firm manufactures two products X and Y. Let the production process involve three departments A, B and C, with time (in minutes) required in each department per unit output of each product given by the table below.

	Department		
	A	B	C
Product X	20	30	45
Product Y	40	30	30

Let x and y be the amounts of X and Y produced each day by the firm. Then the table above states that the amount of time required each day in department A is $20x$ minutes for the production of X and $40y$ minutes for the production of Y, in all $20x+40y$ minutes. Similarly the total amount of time per day required in department B is $30x + 30y$ minutes, and the amount of time per day required in department C is $45x + 30y$ minutes.

Suppose that departments A and B are each available for 8 hours per day, and that department C is available for 11 hours per day. Then each department imposes a constraint on x and y of the form

$$\text{time required} \leq \text{time available.}$$

Using the above data and the fact that there are 60 minutes in each hour, we see that the time constraint for department A is

$$20x + 40y \leq 480.$$

The corresponding constraints for B and C are

$$30x + 30y \leq 480,$$
$$45x + 30y \leq 660.$$

Dividing the first of these three inequalities by 20, the second by 30 and the third by 15, we have

$$x + 2y \leq 24 \quad \text{(constraint A)}$$
$$x + y \leq 16 \quad \text{(constraint B)}$$
$$3x + 2y \leq 44 \quad \text{(constraint C)}$$

Output levels x and y must of course be non-negative, which gives us the additional inequalities

$$x \geq 0, \; y \geq 0.$$

The set of points in the xy–plane satisfying these five inequalities is called the **feasible set** for the firm's production plan.

To sketch the feasible set, we begin by drawing three lines corresponding to the time constraints in the three departments:

$$x + 2y = 24 \quad \text{(line A)}$$
$$x + y = 16 \quad \text{(line B)}$$
$$3x + 2y = 44 \quad \text{(line C)}$$

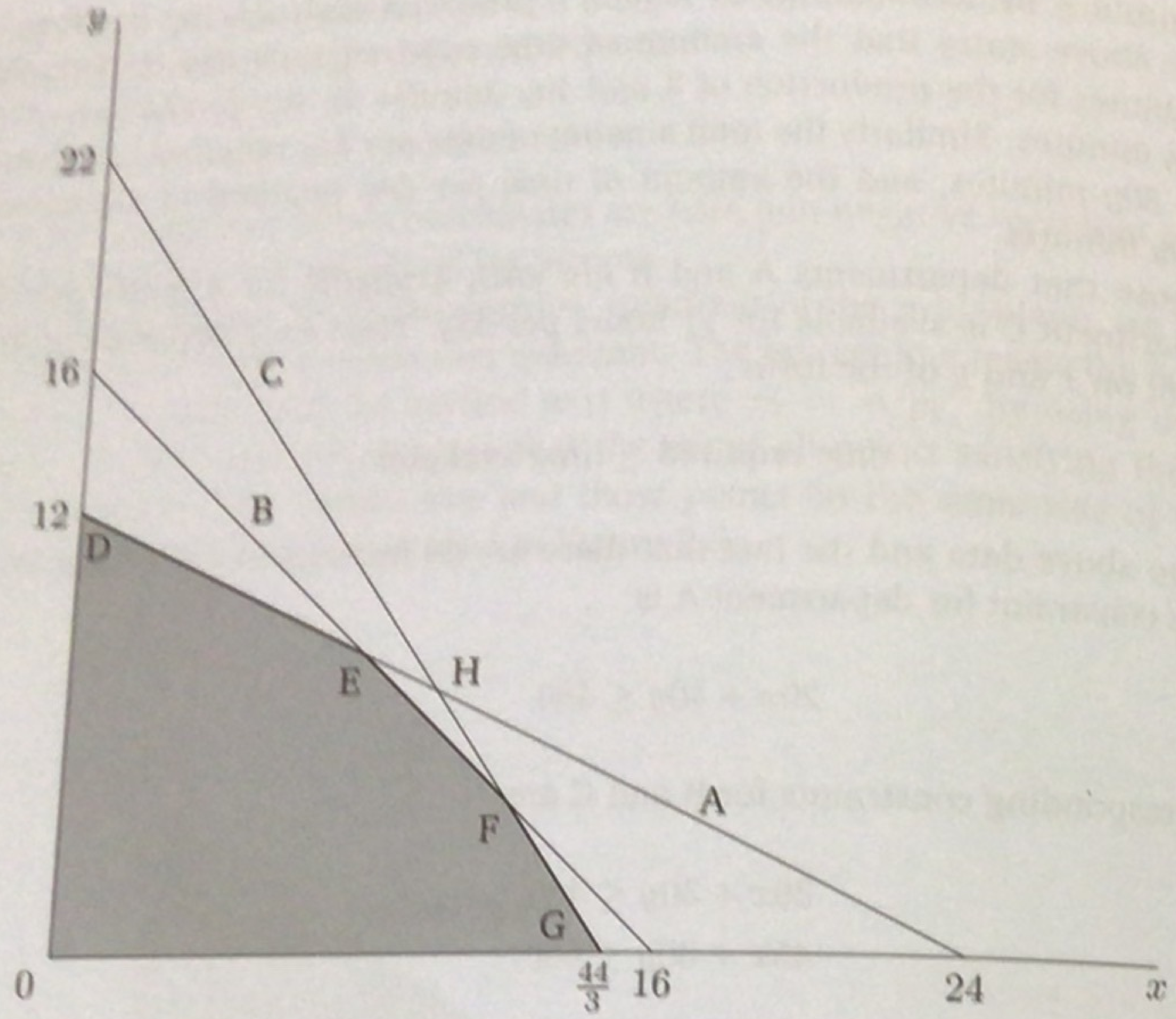

Figure 2.6: **The feasible set for a two-product firm**

Using the origin as test point in the usual way, we see that the set of points satisfying constraint A consist of all points on or to the left of line A. Similar results hold for B and C. Since the feasible set consists of all points which satisfy the three time constraints and have both coordinates non-negative, it is the shaded region ODEFG in Figure 2.6.

The coordinates of the corners of the feasible set are easily found using the methods of Chapter 1. D is the point where line A crosses the y–axis and is therefore the point $(0, 12)$. E is the point of intersection of lines A and B; its coordinates are found by solving simultaneously the equations

$$x + 2y = 24, \quad x + y = 16$$

and turn out to be $(8, 8)$. Similarly, since F is the point of intersection of lines B and C, its coordinates are $(12, 4)$. G is the point where line C crosses the x–axis and is therefore the point $(44/3, 0)$. The entire path DEFG, that part of the boundary of the feasible set which does not coincide with either axis, is called the **constraint boundary**.

Notice finally that the point H where A crosses C is outside the feasible set. The interpretation of this is that the firm cannot use departments A *and* C to full capacity: if it did, it would be producing too much X and Y per day for department B to handle in eight hours.

Exercises

2.2.1 Judy has an income of 10 which she spends on fish and chips, the prices of which are 2 and 3 respectively. Sketch the budget set.

Also sketch the budget set if the prices of the two goods are reversed.

2.2.2 Henry has an income of 18 which he can spend on two goods labelled 1 and 2, with prices 1 and 3 respectively. Sketch the budget set.

Sketch also the budget set in each of the following cases:

(a) income 36, prices 2 and 6 ;

(b) income 90, prices 5 and 15 ;

(c) income 9, prices 0.5 and 1.5 .

What do you notice? Can you formulate a general result?

2.2.3 A firm produces two products X and Y, using a production process involving two departments A and B. The time in minutes required in each department per unit output of each product is given by the following table.

	Department	
	A	B
Product X	16	10
Product Y	8	20

Department A is available for 4 hours per day and department B is available for 5 hours per day. Sketch the feasible set.

2.2.4 Suppose the situation is as in Exercise 2.2.3, but with the additional information that production per unit of X and Y causes the emission of 2 and 3 units of carbon respectively. Sketch the feasible set if total carbon emissions are to be restricted to 48 units per day.

Sketch also the feasible set if total carbon emissions per day are to be restricted to (a) 60 units, (b) 24 units.

2.3 Linear programming

Two of the most important concepts in economic theory are equilibrium and optimisation. Possibly the simplest example of economic equilibrium, clearing of a single market, was discussed in Section 1.2. Optimisation means doing as well as one can subject to given constraints.

To illustrate this, we continue with the example of the two-product firm of the last section, and assume in addition that the firm attempts to maximise profit. Suppose products X and Y yield profits of £30 and £40 per unit respectively. If the firm produces per day x units of X and y of Y, its profit is $30x + 40y$ pounds per day. The firm's problem is to choose the output combination (x, y) which makes profit as large as possible, subject to the constraint that (x, y) be feasible. Recalling from Section 2.2 the inequalities that define our firm's feasible set of output combinations, we may write the profit-maximisation problem as follows:

$$\begin{aligned} \text{maximise} \quad & 30x + 40y \\ \text{subject to} \quad & x + 2y \leq 24 \\ & x + y \leq 16 \\ & 3x + 2y \leq 44 \\ & x \geq 0, \ y \geq 0 \end{aligned}$$

This particular kind of optimisation problem, in which a linear expression is being maximised subject to linear inequalities, is called a **linear maximisation programme**. The expression to be maximised, in this case representing profit, is called the **objective function**.

To solve our maximisation programme we must search over the feasible set to locate the point of maximal profit. We start by considering the different output combinations which yield a given level of profit, say z. If (x, y) is such a combination,

$$30x + 40y = z;$$

the point (x, y) therefore lies on the straight line which cuts the x–axis at $(z/30, 0)$ and the y–axis at $(0, z/40)$. Such a straight line is called an **isoprofit line**.

The isoprofit line just discussed has slope $-3/4$ and intercept $z/40$. Changing the value of profit z gives another isoprofit line with the same slope but a different intercept: note that a higher intercept means a higher level of profit. As we allow z to vary, we obtain what is known as a **family** of isoprofit lines; the lines are parallel to each other, each corresponding to a particular level of profit and with higher lines corresponding to higher profits. Four members of the family are sketched in Figure 2.7, which also reproduces the feasible set from Figure 2.6. Notice that all but one of the four lines meet the feasible set; the exception is IP_4, which corresponds to the highest of four levels of profit.

It is now easy to solve our linear programme. Moving outward from the origin to isoprofit lines corresponding to increasingly high profits, we reach the highest one that meets the feasible set. This is the line IP_3 which meets the feasible set at just one point, namely E. This point gives the profit-maximising output combination. Recalling from Section 2.2 that the coordinates of E are $(8, 8)$, we see that profit is maximised if $x = y = 8$: maximal profit is

$$30 \times 8 + 40 \times 8 = 560.$$

The solution method

Trial-and-error with isoprofit lines is not an efficient method of solving linear programmes, and one feature of Figure 2.7 suggests a better way. We are referring to

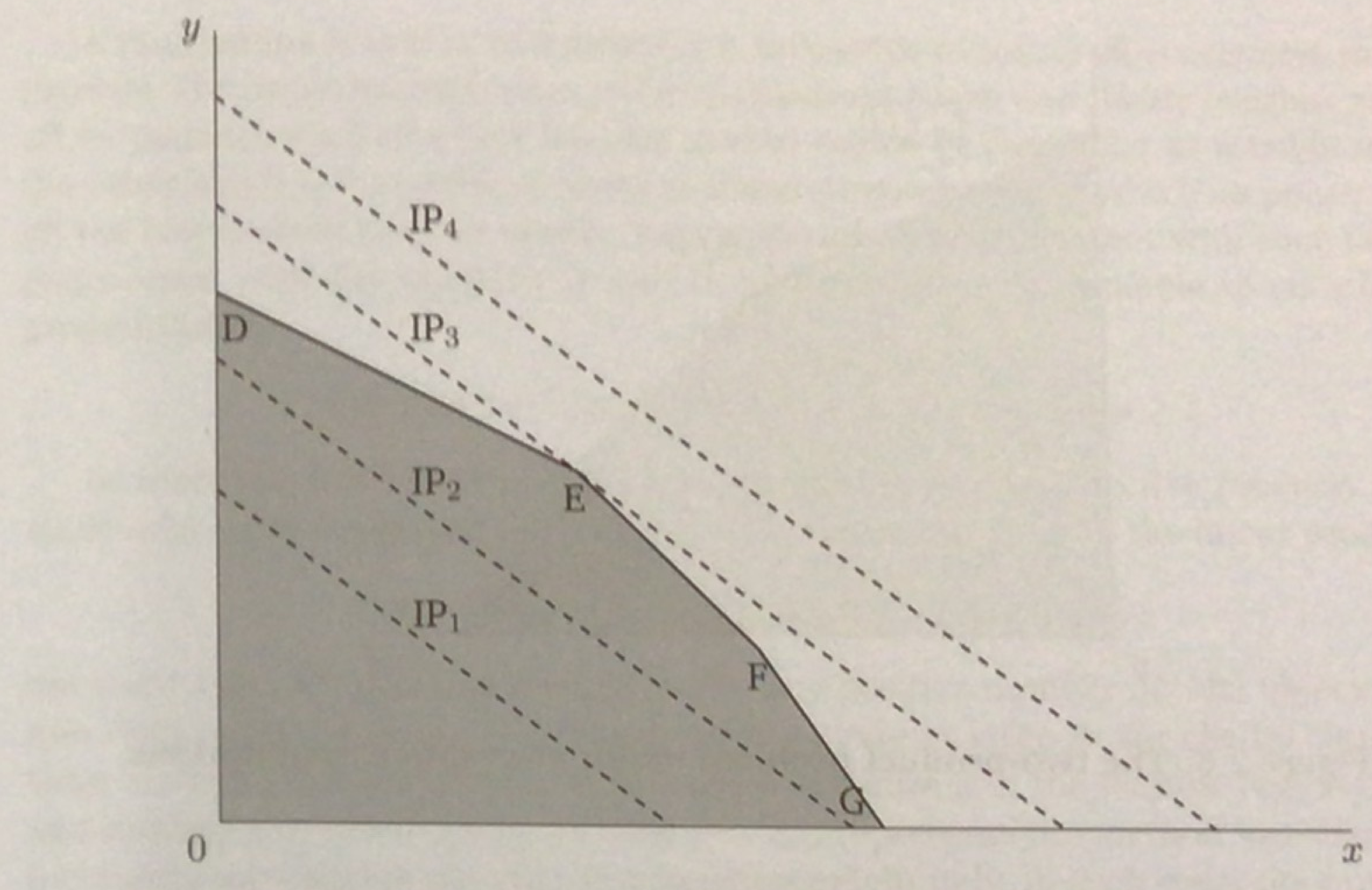

Figure 2.7: **Profit maximisation for a two-product firm**

the fact that at the optimum point E, the slope of the isoprofit line IP_3 lies between the slopes of DE and EF, the two parts of the constraint boundary which meet at E.

Why is this helpful? If you recall how we constructed the feasible set in Section 2.2, you will notice that the slope of each part of the constraint boundary is easily read off from the data of the problem. The part of the boundary from D to E is a segment of what we called line A and therefore has slope $-1/2$; the line-segment EF is part of line B and therefore has slope -1; similarly the slope of FG is $-3/2$. The feasible set is drawn yet again in Figure 2.8; lines are labelled with their slopes, enclosed in square brackets. We know from the data on profits that the slope of each isoprofit line is $-3/4$. To solve the problem, we simply note that $-3/4$ is between $-1/2$ and -1, which locates the optimum point at E.

To summarise, we have solved our profit-maximisation programme by the following method, which is the one we recommend for problems of this type:

1. Sketch the feasible set, calculating the coordinates of its corners and the slopes of its edges (the line-segments that make up the constraint boundary).

2. Calculate the common slope s of the isoprofit lines. It is *not* necessary to sketch these lines.

3. If s lies between the slopes of two adjacent edges, profits are maximised at the corner where these edges meet.

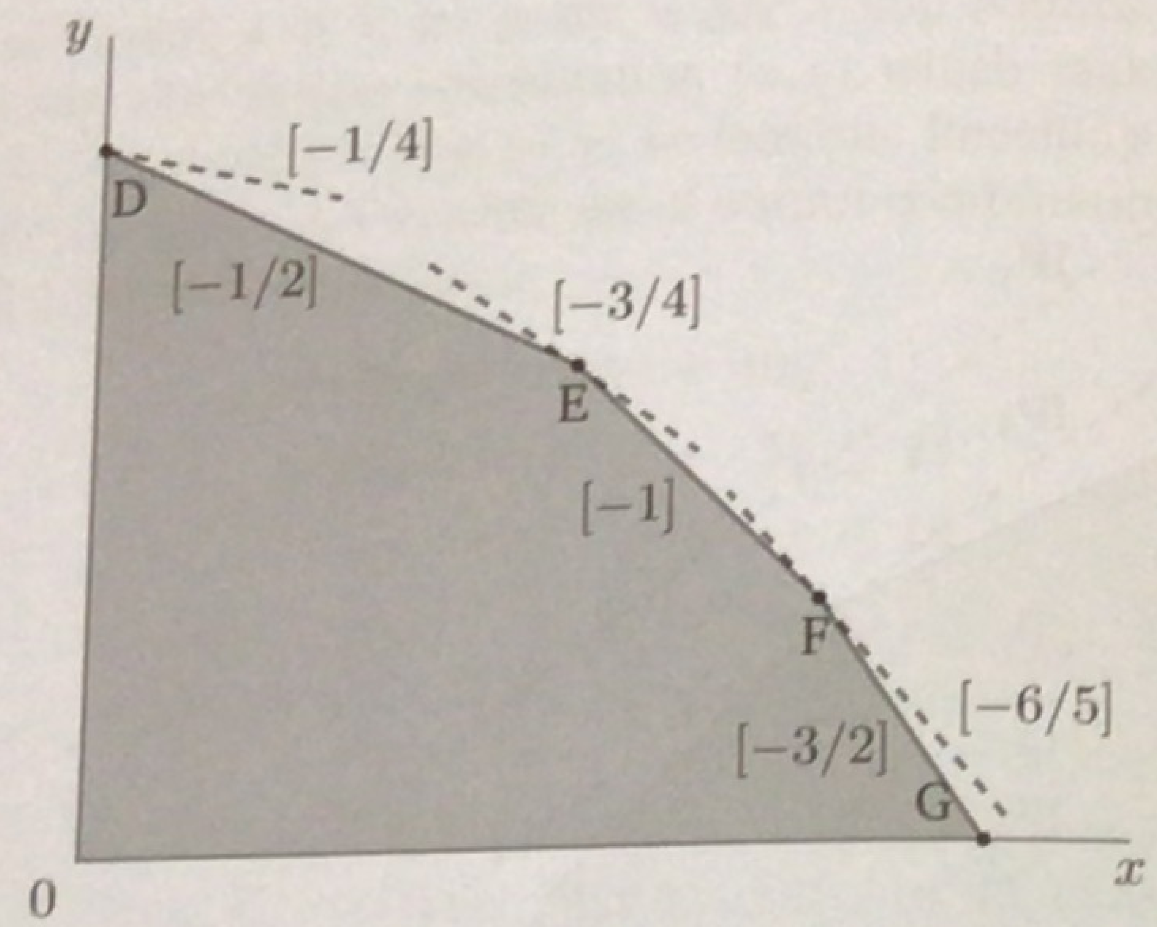

Figure 2.8: **The two-product problem under alternative assumptions**

We now use the recommended method to solve some variants of the problem we have just solved, corresponding to different assumptions about profits; these solutions are also depicted in Figure 2.8. Suppose the unit profits for X and Y are not 30 and 40 as above, but z_X and z_Y respectively. The common slope s of the isoprofit lines is then $-z_X/z_Y$. For example, if $z_X = 60$ and $z_Y = 50$, the slope of each isoprofit line is $-6/5$, which lies between the slopes of EF and FG: hence the optimum is at F. Since F has coordinates $(12, 4)$, maximal profit is 920.

A case of some interest is that where $z_X = 16$ and $z_Y = 54$; here the isoprofit lines are even flatter than DE and the optimum is at the point D on the y–axis; 0 units of X and 12 units of Y are produced, giving a maximal profit of 768. Notice that, in this as in all the other cases, it is the profit *ratio* z_X/z_Y that determines the optimal level of output of each product; the absolute levels of z_X and z_Y serve only to determine the maximal amount of profit.

Another interesting case occurs where $z_X = z_Y = 25$. Here $z_X/z_Y = 1$, so the slope of each isoprofit line is equal to the slope of EF. This implies that the profit-maximisation problem has multiple solutions: any (x, y) combination on the edge EF maximises profits, with a profit of 400.

Generalities and complications

Having analysed the example of a two-product firm in some detail, we are now in a position to make some general points about linear programming. Recall that a linear maximisation programme is an optimisation problem in which a linear expression, called the objective function, is maximised subject to constraints taking the form of linear inequalities. The set of points in the xy–plane which satisfy the constraints is called the feasible set.

A programme is said to be **feasible** if it is possible to satisfy all constraints simultaneously. The profit-maximisation problem discussed above was clearly feasible, as were all its variants, each of which had the shaded region of Figure 2.5 as feasible set. On the other hand, it is possible to think of linear programmes for which *no* points satisfy all the constraints; in other words, the constraints are inconsistent with each other. A programme with this property is said to be **infeasible.** An example of an infeasible programme is

$$\text{maximise } 2x + 3y \text{ subject to } x + y \le 2 \text{ and } x + y \ge 3.$$

In some feasible linear maximisation programmes, the objective function may be made arbitrarily large without violating the constraints. Thus in the linear programme

$$\text{maximise } x + y \text{ subject to } x \ge 2y \text{ and } y \ge 0,$$

the point $(4M, M)$ is in the feasible set for any positive number M; the objective function then takes the value $5M$, which may be made as large as we like by choosing M large enough. This programme is illustrated in Figure 2.9: the feasible region is shaded and upward movement along the line $y = x/4$ represents the method just described of increasing the value of the objective function indefinitely. In such cases the programme is said to be **unbounded**; otherwise the programme is said to be **bounded.**

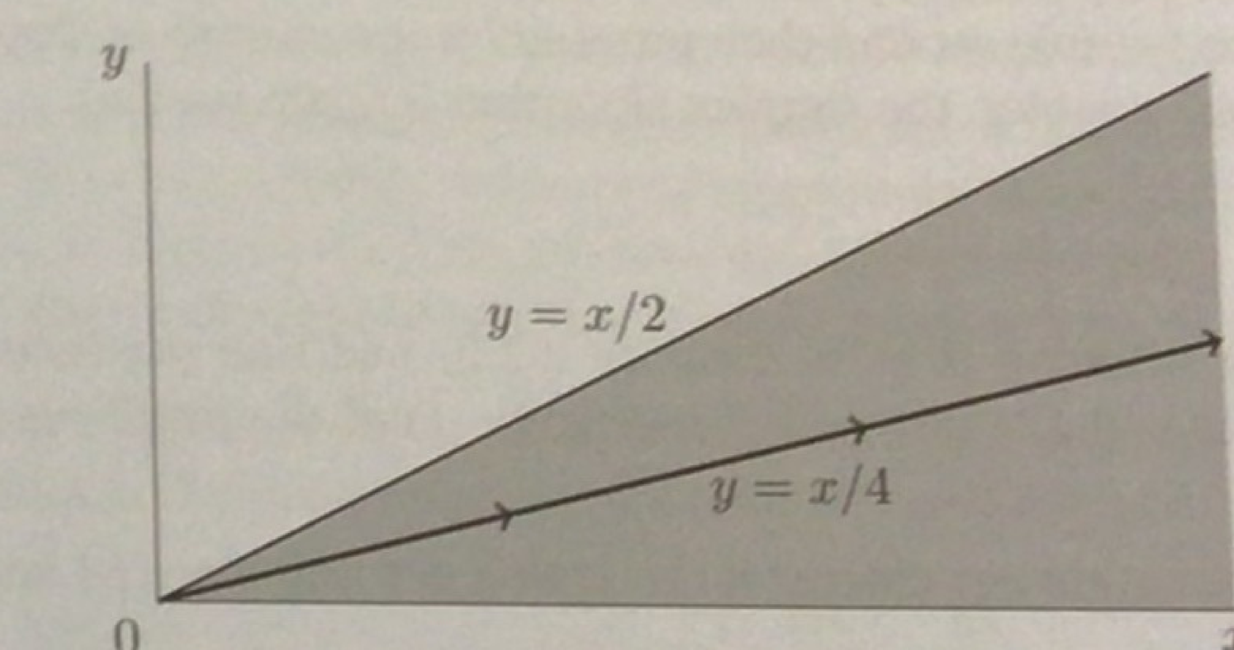

Figure 2.9: **An unbounded linear programme**

The production problem considered earlier in this section is feasible and bounded in all its variants. A linear programme which is feasible and bounded always has at least one solution. If there is a unique solution, it occurs at a corner of the feasible set. If there are multiple solutions, as in our production problem when $z_X = z_Y = 25$, then a solution can still be found at a corner; thus either E or F in Figure 2.8 will do).

The observant reader will have noticed that we have considered only *weak* inequality constraints in this section. In more general linear programming problems, equation constraints may occur, but strict inequality constraints are not allowed. To see the reason for this, consider the innocent-looking problem

$$\text{maximise } x \text{ subject to } x < 1.$$

This problem has no solution: $x = 0.99$ does better than $x = 0.9$, $x = 0.999$ does better still, and so on, but $x = 1$ violates the constraint. To avoid technical difficulties of this kind, linear programming restricts attention to constraints which are equations or weak inequalities. In many problems in economics this is the realistic way to proceed; thus in our production problem it makes sense to assume that the firm can use any of its three departments to maximum availability if it wishes.

Extensions

When there are more than two variables, the graphical approach is no longer possible. It is however possible to generalise the method. The reader should be able to visualise a 'corner' in three dimensions as a point where three planes meet. In dimensions higher than three, no pictorial representation is available but the concept of a corner can still be defined algebraically. The significance of corners is that, if a solution exists, there is a solution at a corner of the feasible set. Thus solution of a linear programme reduces to a search over corners. The most popular way of doing this is known as the simplex algorithm and explained in textbooks on linear programming.

Linear minimisation programmes are just like maximisation programmes, except that we are now minimising a linear expression subject to linear constraints. With two variables, the standard graphical method of solution is as for maximisation: we carefully draw the feasible set and then proceed by comparison of slopes. When there are more than two variables, the simplex algorithm is again needed.

Exercises

2.3.1 Suppose the situation is as in Exercise 2.2.3, and that products X and Y yield profits of £12 and £16 per unit respectively. Find the profit-maximising output levels.

How would your answer change if the profits per unit were £4 and £1?

2.3.2 Suppose the situation is as in Exercise 2.2.4, and that products X and Y yield profits of £12 and £16 per unit respectively. Find the profit-maximising output levels when total carbon emissions are restricted to 48 units per day.

How would your answer change if permitted carbon emissions were to fall to 24 units per day?

Problems on Chapter 2

2–1. The following is a general version of the macroeconomic model of Problem 1–3. As before, the unknowns are Y (national income), C (consumption) and T (tax collection); I (investment) and G (government expenditure) are assumed to be known. The equations of the model are

$$Y = C + I + G, \quad C = c_0 + c_1(Y - T), \quad T = t_0 + t_1 Y;$$

c_0, c_1, t_0, t_1 are constant parameters, with $0 < c_1 < 1$ and $0 < t_1 < 1$. Find Y, C and T in terms of I, G and the parameters. What happens to Y, C and T if G increases by x units? If $x > 0$, can we be sure that Y increases by more than x units? can we be sure that C increases by less than x units?

2–2. Consider again consumer Ian of Section 2.2, who has income m and consumes two goods labelled 1 and 2. Assume as before that good 2 has price p_2. Good 1 is priced at p_1 up to consumption level z, and at $p_1 + t$ per unit on consumption in excess of z, where $t > 0$. Sketch the budget set.

Also sketch the budget set

(i) when $t < 0$;

(ii) when consumption of good 1 is rationed at z.

2–3. Consider again the economy of Exercise 1.4.2 and Problem 1–4. Suppose that, in addition to the produced goods X and Y, there are two kinds of non-produced input, labour and land. Assume that production of each unit of gross output of X requires the use of 7 units of labour and 3 units of land, and production of each unit of gross output of Y requires the use of 6 units of labour and 2 units of land.

(i) Using your answer to Problem 1–4, calculate total usage of labour and land when the *net* outputs of X and Y are a and b respectively.

(ii) Suppose the the economy's total endowments of labour and land are 800 and 300 units respectively. What conditions must be satisfied by the numbers a and b if the economy can produce net outputs of a units of X and b units of Y? Draw a diagram illustrating feasible combinations of a and b.

2–4. The diet of Oleg the Russian Blue consists of only two food items, FishBits and KittyCrackers. Prices and nutrient contents per unit are as follows.

	FB	KC
Price	2	1
Calcium	10	4
Protein	5	5
Calories	2	6

Oleg has minimum daily requirements of 20 units of calcium, 20 units of protein and 12 units of calories. Find the combination of the two food items which will satisfy his daily requirements and entail least cost. State the least cost.

How does the solution change if the prices change to

(i) 4 and 2, (ii) 3 and 2, (iii) 3 and 1?

Find a value of the price ratio of the two food items for which the solution is not unique. Is there more than one such value?