Funcions (II) Potencials, desigualtats, inequacions

Projecte Math21, Departament Economia i Empresa, UPF

25 de març de 2021

Índex

Aquest és un document de treball INTERN, en fase de discussió i molt preliminar. No en feu difussió, sisplau.

(Enllaç al document principal)
(Enllaç a la versió pdf d'aquest document)
(Enllaç a la font LaTeX)
(Enllaç als fitxers de les figures)

1 Objectius d'aprenentatge

- Composició de funcions, funció inversa

- Funcions potencials, $y = ax^b$ comportament segons els valors de a, b
- Corbes d'isonivell
- Desigualtats, inequacions
- Encara transformacions de gràfiques

2 Prerequisits

3 Guia pel professor

Es tracta d'ampliar el que s'ha vist al tema I-2 afegint conceptes importants com la composició, la inversa.

Les funcions potencials apareixen constantment en el model econòmics que estan estudiant en paralel a IntroMicro. En particular, les corbes de isobenefici i similars. Per això convé introduir ja la idea de corba de nivell sense entrar gaire en les funcions de dues variables.

3.1 Presentació del tema

3.2 Materials bàsics

4 Activitats autònomes

4.0.1 Videos sobre composició de funcions

De la Kahn Academy:

Introduction to function composition — Functions and their graphs — Algebra II — Khan Academy (6min)

Evaluating composite functions example — Functions and their graphs — Algebra II — Khan Academy (4min)

Creating new function from composition — Functions and their graphs — Algebra II — Khan Academy (3 min)

Sobre funció inversa: Introduction to function inverses — Functions and their graphs — Algebra II — Khan Academy (9 min)

Després de veure aquests videos, respón aquestes qüestions:

QUIZZ
Si $f(x) = |x|$ i $g(x) = x - 2$ i posem $h = f \circ g$ (ho llegim f *després de* g) aleshores
$==h(x) = |x - 2|$
$= h(x) = |x| - 2$
= qualsevulla de les dues pot ser veritat.

4.0.2 Funcions compostes

Aquest seria un tema ideal per fer-lo treballar autònomament. Si no es té accés al llibre An$Appr, caldria fer-ne una versió pròpia, en Mates I començariem amb un nivell com aquest.

Llegeix i practica les pàgines 24-28 del llibre IB-HL-AnAppr:

1.3 Composite functions

Composition of functions

The **argument** of a function is the variable or expression on which a function operates. For example, the argument of $f(x) = x^3$ is x, the argument of $g(x) = \sqrt{x-3}$ is $x - 3$, and the argument of $y = 10^{2x}$ is $2x$.

Consider the function in Example 1.16, $f(x) = \sqrt{x+4}$. When we evaluate $f(x)$ for a certain value of x in the domain, for example, $x = 5$, it is necessary to perform computations in two separate steps in a certain order.

$f(5) = \sqrt{5+4} \Rightarrow f(5) = \sqrt{9}$ Step 1: compute the sum of 5 + 4

$\Rightarrow f(5) = 3$ Step 2: compute the square root of 9

Given that the function has two separate evaluation steps, $f(x)$ can be seen as a combination of two simpler functions that are performed in a specified order. According to how $f(x)$ is evaluated, the simpler function to be performed first is the rule of adding 4 and the second is the rule of taking the square root. If $h(x) = x + 4$ and $g(x) = \sqrt{x}$, then we can create (compose) the function $f(x)$ from a combination of $h(x)$ and $g(x)$ as follows:

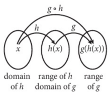

Figure 1.19 Mapping for composite function $g(h(x))$

$f(x) = g(h(x))$

$= g(x+4)$ Step 1: substitute $x + 4$ for $h(x)$ making $x + 4$ the argument of $g(x)$

$= \sqrt{x+4}$ Step 2: apply the function $g(x)$ on the argument $x + 4$

We obtain the rule $\sqrt{x+4}$ by first applying the rule $x + 4$ and then applying the rule $\sqrt{x}$. A function that is obtained from simpler functions by applying one after another in this way is called a **composite function**. $f(x) = \sqrt{x+4}$ is the **composition** of $h(x) = x + 4$ followed by $g(x) = \sqrt{x}$. In other words, f is obtained by substituting h into g, and can be denoted in function notation by $g(h(x))$ – read 'g of h of x.'

The composition of two functions, g and h, such that h is applied first and g second is given by $(g \circ h)(x) = g(h(x))$. The domain of the composite function $g \circ h$ is the set of all x in the domain of h such that $h(x)$ is in the domain of g.

Start with a number x in the domain of h and find its image $h(x)$. If this number $h(x)$ is in the domain of g, we then compute the value of $g(h(x))$. The resulting composite function is denoted as $(g \circ h(x))$. See Figure 1.19.

Example 1.18

If $f(x) = 3x$ and $g(x) = 2x - 6$, find:

(a) (i) $(f \circ g)(5)$ (ii) Express $(f \circ g)(x)$ as a single function rule (expression).

(b) (i) $(g \circ f)(5)$ (ii) Express $(g \circ f)(x)$ as a single function rule (expression).

(c) (i) $(g \circ g)(5)$ (ii) Express $(g \circ g)(x)$ as a single function rule (expression).

1 Algebra and function basics

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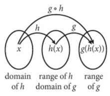

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(c) (i) $(g \circ g)(5)$ (ii) Express $(g \circ g)(x)$ as a single function rule (expression).

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Solution

(a) (i) $(f \circ g)(5) = f(g(5)) = f(2 \cdot 5 - 6) = f(4) = 3 \cdot 4 = 12$

(ii) $(f \circ g)(x) = f(g(x)) = f(2x - 6) = 3(2x - 6) = 6x - 18$

Therefore, $(f \circ g)(x) = 6x - 18$

Check with result from (i): $(f \circ g)(5) = 6 \cdot 5 - 18 = 30 - 18 = 12$

(b) (i) $(g \circ f)(5) = g(f(5)) = g(3 \cdot 5) = g(15) = 2 \cdot 15 - 6 = 24$

(ii) $(g \circ f)(x) = g(f(x)) = g(3x) = 2(3x) - 6 = 6x - 6$

Therefore, $(g \circ f)(x) = 6x - 6$

Check with result from (i):$(g \circ f)(5) = 6 \cdot 5 - 6 = 30 - 6 = 24$

(c) (i) $(g \circ g)(5) = g(g(5)) = g(2 \cdot 5 - 6) = g(4) = 2 \cdot 4 - 6 = 2$

(ii) $(g \circ g)(x) = g(g(x)) = g(2x - 6) = 2(2x - 6) - 6 = 4x - 18$

Therefore, $(g \circ g)(x) = 4x - 18$

Check with result from (i): $(g \circ g)(5) = 4 \cdot 5 - 18 = 20 - 18 = 2$

It is important to notice that in parts (a)(ii) and (b)(ii) in Example 1.18, $f \circ g$ is not equal to $g \circ f$. At the start of this section, it was shown how the two functions $h(x) = x + 4$ and $g(x) = \sqrt{x}$ could be combined into the composite function $(g \circ h)(x)$ to create the single function $f(x) = \sqrt{x+4}$. However, the composite function $(h \circ g)(x)$ (the functions applied in reverse order) creates a different function: $(h \circ g)(x) = h(g(x)) = h(\sqrt{x}) = \sqrt{x} + 4$. Since, $\sqrt{x} + 4 \neq \sqrt{x+4}$ then $f \circ g$ is not equal to $g \circ f$. Is it always true that $f \circ g \neq g \circ f$? The next example will answer that question.

Example 1.19

Given $f: x \mapsto 3x - 6$ and $g: x \mapsto \frac{1}{3}x + 2$, find:

(a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$

Solution

(a) $(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{3}x + 2\right) = 3\left(\frac{1}{3}x + 2\right) - 6 = x + 6 - 6 = x$

(b) $(g \circ f)(x) = g(f(x)) = g(3x - 6) = \frac{1}{3}(3x - 6) + 2 = x - 2 + 2 = x$

Example 1.19 shows that it is possible for $f \circ g$ to be equal to $g \circ f$. You will learn in the next section that this occurs in some cases where there is a special relationship between the pair of functions. However, in general $f \circ g \neq g \circ f$.

Decomposing a composite function

In examples 1.18 and 1.19, we created a single function by forming the composite of two functions. As with the function $f(x) = \sqrt{x+4}$, it is also important for us to be able to identify two functions that make up a composite function, in other words, to decompose a function into two simpler functions. When we are doing this it is very useful to think of the function that is applied first as the inside function, and the function that is applied second as the outside function. In the function $f(x) = \sqrt{x+4}$, the inside function is $h(x) = x + 4$ and the outside function is $g(x) = \sqrt{x}$.

Example 1.20

Each of these functions is a composite function of the form $(f \circ g)(x)$. For each, find the two component functions f and g.

(a) $h: x \mapsto \frac{1}{x+3}$ (b) $k: x \mapsto 2^{4x+1}$ (c) $p(x) = \sqrt[3]{x^2 - 4}$

Solution

(a) When we evaluate the function $h(x)$ for a certain x in the domain, we first evaluate the expression $x + 3$, and then evaluate the expression $\frac{1}{x}$. Hence, the inside function (applied first) is $y = x + 3$, and the outside function (applied second) is $y = \frac{1}{x}$. So the two component functions are $g(x) = x + 3$ and $f(x) = \frac{1}{x}$

(b) Evaluating $k(x)$ requires us to first evaluate the expression $4x + 1$, and then evaluate the expression 2^x. Hence, the inside function is $y = 4x + 1$, and the outside function is $y = 2^x$. The two composite functions are $g(x) = 4x + 1$ and $f(x) = 2^x$.

(c) Evaluating $p(x)$ requires us to perform three separate evaluation steps: squaring a number, subtracting four, and then taking the cube root. Hence, it is possible to decompose $p(x)$ into three component functions: $h(x) = x^2$, $g(x) = x - 4$ and $f(x) = \sqrt[3]{x}$. However, for our purposes it is best to decompose the composite function into only two component functions: $g(x) = x^2 - 4$, and $f(x) = \sqrt[3]{x}$.

Finding the domain of a composite function

It is important to note that in order for a value of x to be in the domain of the composite function $g \circ h$, two conditions must be met: (1) x must be in the domain of h, and (2) $h(x)$ must be in the domain of g. Likewise, it is also worth noting that $g(h(x))$ is in the range of $g \circ h$ only if x is in the domain of $g \circ h$. The next example illustrates these points – and also that, in general, the domains of $g \circ h$ and $h \circ g$ are not the same.

Example 1.21

Let $g(x) = x^2 - 4$ and $h(x) = \sqrt{x}$. Find:

(a) $(g \circ h)(x)$ and its domain and range

(b) $(h \circ g)(x)$ and its domain and range.

Solution

First, establish the domain and range for both g and h. For $g(x) = x^2 - 4$, the domain is $x \in \mathbb{R}$ and the range is $y \geqslant -4$. For $h(x) = \sqrt{x}$, the domain is $x \geqslant 0$ and the range is $y \geqslant 0$.

(a) $(g \circ h)(x) = g(h(x))$

$= g(\sqrt{x})$ To be in the domain of $g \circ h$, $\sqrt{x}$ must be defined for $x \Rightarrow x \geqslant 0$

$= (\sqrt{x^2}) - 4$ Therefore, the domain of $g \circ h$ is $x \geqslant 0$

$= x - 4$ Since $x \geqslant 0$, then the range for $y = x - 4$ is $y \geqslant -4$.

Therefore, $(g \circ h)(x) = x - 4$, and its domain is $x \geqslant 0$, and its range is $y \geqslant -4$

(b) $(h \circ g)(x) = h(g(x))$ $g(x) = x^2 - 4$ must be in the domain of $h \Rightarrow x^2 - 4 \geqslant 0 \Rightarrow x^2 \geqslant 4$

$= h(x^2 - 4)$ Therefore, the domain of $h \circ g$ is $x \leqslant -2$ or $x \geqslant 2$ and

$= \sqrt{x^2 - 4}$ with $x \leqslant -2$ or $x \geqslant 2$, the range for $y = \sqrt{x^2 - 4}$ is $y \geqslant 0$

Therefore, $(h \circ g)(x) = \sqrt{x^2 - 4}$, and its domain is $x \leqslant -2$ or $x \geqslant 2$, and its range is $y \geqslant 0$

Exercise 1.3

1. Let $f(x) = 2x$ and $g(x) = \frac{1}{x-3}$, $x \neq 0$

Find the value of:

(a) $(f \circ g)(5)$ **(b)** $(g \circ f)(5)$

Find the function rule (expression) for:

(c) $(f \circ g)(x)$ **(d)** $(g \circ f)(x)$

2. Let $f: x \mapsto 2x - 3$ and $g: x \mapsto 2 - x^2$

Evaluate:

(a) $(f \circ g)(0)$ **(b)** $(g \circ f)(0)$ **(c)** $(f \circ f)(4)$

(d) $(g \circ g)(-3)$ **(e)** $(f \circ g)(-1)$ **(f)** $(g \circ f)(-3)$

Find the expression for:

(g) $(f \circ g)(x)$ **(h)** $(g \circ f)(x)$ **(i)** $(f \circ f)(x)$ **(j)** $(g \circ g)(x)$

3. For each pair of functions, find $(f \circ g)(x)$ and $(g \circ f)$ domain for each.

(a) $f(x) = 4x - 1$, $g(x) = 2 + 3x$ **(b)** $f(x) = x^2$

(c) $f(x) = \sqrt{x+1}$, $g(x) = 1 + x^2$ **(d)** $f(x) =$

(e) $f(x) = 3x + 5$, $g(x) = \frac{x-5}{3}$ **(f)** $f(x) = 2 -$

(g) $f(x) = \frac{2x}{4-x}$, $g(x) = \frac{1}{x^2}$

(h) $f(x) = \frac{2}{x+3} - 3$, $g(x) = \frac{2}{x+3} - 3$

(i) $f(x) = \frac{x}{x-1}$, $g(x) = x^2 - 1$

4. Let $g(x) = \sqrt{x-1}$ and $h(x) = 10 - x^2$. Find:

(a) $(g \circ h)(x)$ and its domain and range

(b) $(h \circ g)(x)$ and its domain and range.

5. Let $f(x) = \frac{1}{x}$ and $g(x) = 10 - x^2$. Find:

(a) $(f \circ g)(x)$ and its domain and range

(b) $(g \circ f)(x)$ and its domain and range.

6. Determine functions g and h so that $f(x) = g(h(x))$

(a) $f(x) = (x+3)^2$ **(b)** $f(x) = \sqrt{x-5}$

(d) $f(x) = \frac{1}{x+3}$ **(e)** $f(x) = 10^{x+1}$

(g) $f(x) = |x^2 - 9|$ **(h)** $f(x) = \frac{1}{\sqrt{x-5}}$

7. Find the domain for:

(i) the function f **(ii)** the function g **(iii)** the co

(a) $f(x) = \sqrt{x}$, $g(x) = x^2 + 1$ **(b)** $f(x) = \frac{1}{x}$,

(c) $f(x) = \frac{3}{x^2 - 1}$, $g(x) = x + 1$ **(d)** $f(x) = 2x$

1.4 Inverse functions

Pairs of inverse functions

If we choose a number and cube it (raise it to the power the cube root of the result, the answer is the original num would occur if we applied the two rules in the reverse or the cube root of a number and then cube the result; agai original number.

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Exercise 1.3

1. **(a)** $(f \circ g)(5) = f(g(5)) = f(\frac{1}{5-3}) = f(\frac{1}{2}) = 2 \cdot \frac{1}{2} = 1$

(b) $(g \circ f)(5) = g(f(5)) = g(2 \cdot 5) = g(10) = \frac{1}{10-3} = \frac{1}{7}$

(c) $(f \circ g)(x) = f(g(x)) = f(\frac{1}{x-3}) = 2 \cdot \frac{1}{x-3} = \frac{2}{x-3}$

(d) $(g \circ f)(x) = g(f(x)) = g(2x) = \frac{1}{2x-3}$

2. **(a)** $(f \circ g)(0) = f(g(0)) = f(2-0^2) = f(2) = 2 \cdot 2 - 3 = 1$

(b) $(g \circ f)(0) = g(f(0)) = g(2 \cdot 0 - 3) = g(-3) = 2 - (-3)^2 = 2 - 9 = -7$

(c) $(f \circ f)(4) = f(f(4)) = f(2 \cdot 4 - 3) = f(5) = 2 \cdot 5 - 3 = 7$

(d) $(g \circ g)(-3) = g(g(-3)) = g(2-(-3)^2) = g(-7) = 2 - (-3)^2 = 2 - 49 = -47$

(e) $(f \circ g)(-1) = f(g(-1)) = f(2-(-1)^2) = f(1) = 2 \cdot 1 - 3 = -1$

(f) $(g \circ f)(-3) = g(f(-3)) = g(2 \cdot (-3) - 3) = g(-9) = 2 - (-9)^2 = 2 - 81 = -79$

(g) $(f \circ g)(x) = f(g(x)) = f(2-x^2) = 2 \cdot (2-x^2) - 3 = 4 - 2x^2 - 3 = 1 - 2x^2$

(h) $(g \circ f)(x) = g(f(x)) = g(2x-3) = 2 - (2x-3)^2$

$= 2 - (4x^2 - 12x + 9) = 2 - 4x^2 + 12x - 9$

$\Rightarrow (g \circ f)(x) = -4x^2 + 12x - 7$

(i) $(f \circ f)(x) = f(f(x)) = f(2x-3) = 2(2x-3) - 3 = 4x - 6 - 3 = 4x - 9$

(j) $(g \circ g)(x) = g(g(x)) = g(2-x^2) = g(2-(2-x^2)^2)$

$= 2 - (4 - 4x^2 + x^4) = 2 - 4 + 4x^2 - x^4$

$\Rightarrow (g \circ g)(x) = -x^4 + 4x^2 - 2$

3. In each question, when finding the domain of $f \circ g$, check the following two conditions:

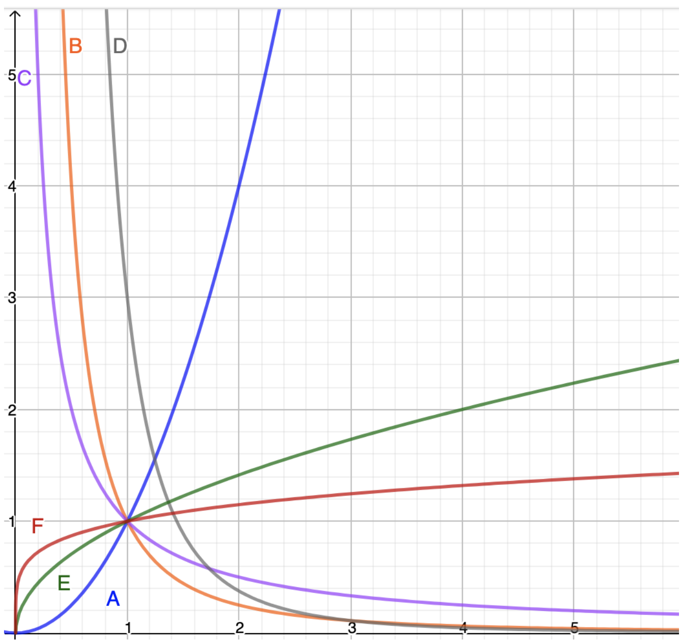
- the input x is in the domain of g, because the first rule to be applied to x is g, the inside function of $f(g(x))$.
- the output $g(x)$ is in the domain of f (the range of g must be either equal to or a subset of the domain of f).

The same applies when finding the domain of $g \circ f$: x must be in the domain of f, because the first rule to be applied to x is f, the inside function of $g(f(x))$, and $f(x)$ is in the domain of g.

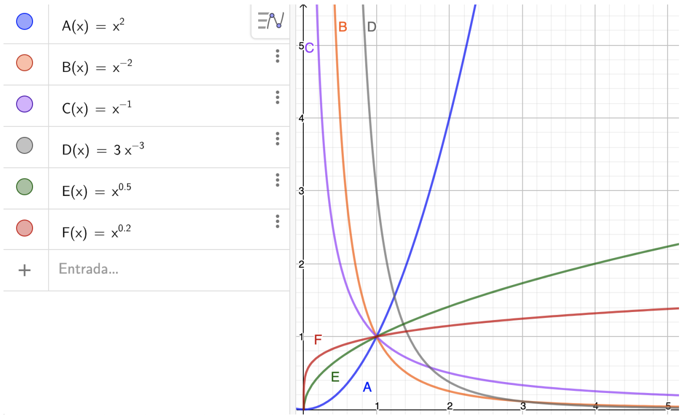
Les solucions:

4.0.3 Funcions potencials

Fes correspondre cada gràfica a cadascuna de les funcions potencials que es llisten:



$$F1 : 3x^{-3}; \quad F2 : x^{-1}; \quad F3 : x^{0.5}; \quad F4 : x^{-2}; \quad F5 : x^{0.2}; \quad F6 : x^2$$

Solució: 

4.0.4 Desigualtats amb potencials

Prenem aquests fet sense demostració:

Fet 1 Si $x < y$ i $a > 0$ aleshores $ax < ay$.

Fet 2 Si $x < y$ i $a < 0$ aleshores $ax > ay$.

Fet 3 Si $x < y$, aleshores $x^{-1} > y^{-1}$

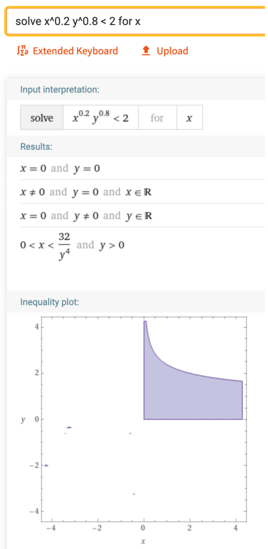
A partir d'això, podem per exemple, demostrar que
Fet 4: Si $0 < x < y$ i $0 < a < b$ aleshores $ax < by$.
Demostració
Multiplicant $0 < x < y$ per a i $0 < a < b$ per y, obtenim $ax < ay$ i $ay < by$ del que es desprén que $ax < by$. □

Ara podem fer servir el Fet 4 per demostrar que
Fet 5: Si $0 < x < y$ aleshores $x^2 < y^2$ i també $x^n < y^n$ per qualsevol nombre natural $n > 0$. La demostració és immediata, només cal multiplicar la primera desigualtat del Fet 4 per ella mateixa, i per demostrar la segona part caldrà multiplicar-la per ella mateixa n vegades. □

El Fet 3 es pot veure clar gràficament: la gràfica de la funció $y = 1/x$ és sempre decreixent, és a dir, si ens desplacem d'esquerra a dreta , els valors de y creixents, la gràfica "fa sempre baixada". Comprova-ho ara fent la gràfica amb Geogebra o similar.

Aquest Fet 3 es pot demostrar per contradicció: si fos cert que $x < y$ i que $x^{-1} < y^{-1}$, multiplicant les dues expressions i segons el Fet 4, tindriem $1 < 1$ que és ben fals!

Wolfram Alpha resol les inequacions, tot i que de vegades el resultat que dona pot ser difícil d'interpretar. Com interpretes aquest resultat? Intenta resoldre tu la ineqüació



$$x^{0.2}y^{0.8} < 2$$

i comprova-ho.

4.0.5 Una funció de producció potencial

De fet aquí R és una micaprematur, si estem a la setmana 3 del curs potser encara no l'han vist gaire.

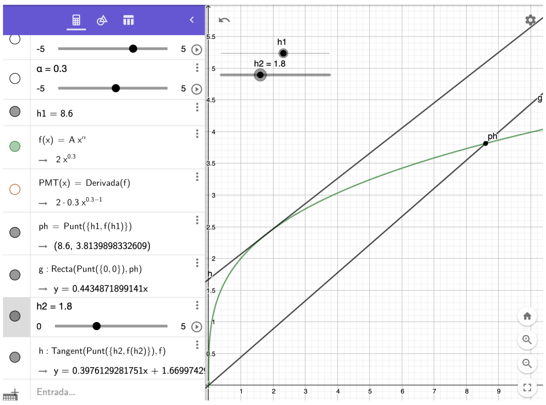
En el tema 3.1 Trabajo y producción del llibre de text de Introducció a la macroeconomia heu vist un exemple de funció de producció del tipus $y = f(h) = Ah^{\alpha}$ amb $A > 0$ i $0 < \alpha < 1$.

Dibuixa amb Geogebra o amb R el gràfic d'aquesta funció per diferents valors de A i de α.

Amb R pots fer servir aquest codi (copia'l a la finestra del R Script per poder modificar-lo)

```
A <- 2
alpha <- 0.3
f <- function(h) A*h^alpha
plot.function(f, from=0, to=15)
PMT <- function(h) A*alpha*h^(alpha-1)
PMeT <- function(h) f(h)/h
h1 <- 10
# per dibuixar la recta del producte mitja del treball per h1
abline(0, PMeT(h1))
# per dibuixar la tangent en el punt d'abcisa h2
h2 <- 4
abline(a=f(h2) - h2*PMT(h2), b=PMT(h2))
```

En Geogebra, hauries de ser capaç de reproduir això:



(solució: vegeu https://www.geogebra.org/calculator/z5gbxwvt)

QUIZZ
En R, fem servir la ordre `abline(a, b)`
= per dibuixar la recta que passa per a i per b
= per dibuixar la recta de pendent b i que talla l'eix horitzontal per $x = a$
== per dibuixar la recta de pendent b i que talla l'eix vertical per $y = a$
= cap de les altres

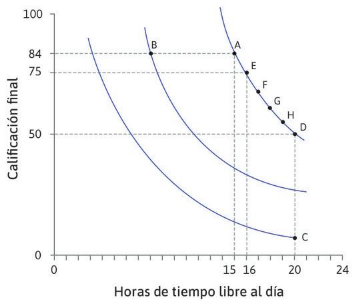
QUIZZ
Per dibuixar el producte mitjà del treball fem servir la ordre `Derivada`
= tant en R com en Geogebra
== en Geogrbra, però no en R
=en R però no en Geogebra
= en cap dels dos.

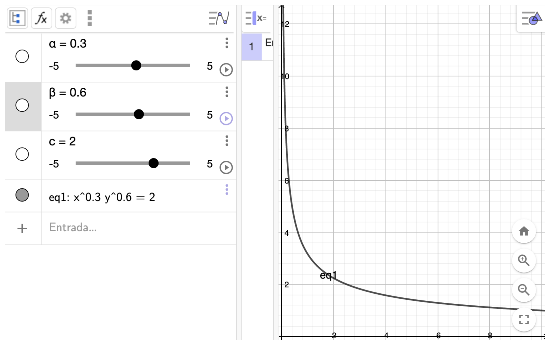
QUIZZ
La funció de producció que planteja el *Leibniz* que has llegit és tal que si Alexei dedica moltíssimes hores de treball, mai aconseguirà superar la qualificació de 9. La funció que fa servir després en l'exemple
= s'assembla a la desitjada però permetria que Alexei obtingués més de 20 si hi dediqués unes 300 hores
== per obtenir un 10 hauria de dedicar-hi unes 214 hores
= per resoldre l'equació $f(h) = 10$ cal fer servir logaritmes.

4.0.6 Corbes de isonivell

Sovint tenim dues quantitats x, y lligades per una funció $y = f(x)$ però que també ens donen una tercera quantitat $z = f(x, y)$, és a dir, que donats valors de x, y podem calular el valor z associat. Això ho esteu trobant en exemples aplicats a la microeconomia.

Per exemple, refresca el capítol 3.2 Preferencias i en concret el *Leibniz* sobre les corbes d'indiferencia on es considera la funció d'utilitat $U(t, y)$ on t són les hores de temps lliure d'Alexei, i y és la nota de l'examen que obté i que depèn de les hores que no dediqui a temps lliure: $y = y(t)$.





La formulació matemàtica que proposen: $U(t, y) = t^\alpha y^\beta$ (és una funció tipus Cobb-Douglas).

Les corbes d'indiferència $U(t, y) = c$ són els punts on la utilitat és la mateixa, c, en diem també corbes de nivell o corbes de isonivell. En la figura pots veure com visualitzar les corbes d'isonivell en Geogebra.

Si tenim $U(t, y) = c$, com que $U(t, y) = t^\alpha y^\beta$, podem expressar y en funció de t. Per aïllar la y en $t^\alpha y^\beta = c$ dividim a le dues bandes per t^α i tenim $y^\beta = ct^{-\alpha}$ i si ara elevem els dos termes de la igualtat a $1/\beta$, tenim

$$\left(y^\beta\right)^{1/\beta} = \left(ct^{-\alpha}\right)^{1/\beta}, \quad \text{és a dir,} \quad y = c^{-\alpha} t^{-\alpha/\beta}$$

QUIZZ
Aplica això al cas $U(t, y) = \sqrt{ty}$.
Obtindrem:
== $y = c^2 t^{-1}$
= $y = ct^{-1}$
= $y = ct^{-2}$
= cap de les anteriors

Tornarem a aquest exemple més endavant per discutir aspectes associats amb les derivades.

5 Llista d'exercicis

1. Si $f(x) = ax + b$ i $g(x) = cx + d$ i es compleix $f \circ g = g \circ f$, demostra que $(a - 1)d = (c - 1)b$.
2. Si $f(x) = x^a$, calcula i simplifica $f(f(x))$.
3. Si $f(x) = x^a$, calcula i simplifica $f^{-1}(x)$.
4. Si $f(x) = x^a$ i $g(x) = x^b$ demostra que es compleix $f \circ g = g \circ f$.

5.0.1 Desigualtats amb potencials

1. Fent servir els fets expressats a 4.0.4, demostra que:
   Si $0 < x < y$, per tota $n > 1$, $x^{-n} > y^{-n}$
2. Explica, a partir de la seva gràfica, si la funció $y = x^a$ per $x > 0$ i amb $a > 0$ és creixent o decreixent segons siguin els valors de a.
3. En el context i notacions de 4.0.6 i suposant $t, y > 0$ com és habitual en Micoreconomia, aïlla la t de la inequació $U(t, y) < c$. Tingues bé en compte els fets sobre desigualtats i potències que hem vist abans (4.0.4)

5.0.2 Exercici

6 Suplements avançats

7 Exercicis per exàmens