Exercise 1.3

1. **(a)** $(f \circ g)(5) = f(g(5)) = f(\frac{1}{5-3}) = f(\frac{1}{2}) = 2 \cdot \frac{1}{2} = 1$

(b) $(g \circ f)(5) = g(f(5)) = g(2 \cdot 5) = g(10) = \frac{1}{10-3} = \frac{1}{7}$

(c) $(f \circ g)(x) = f(g(x)) = f(\frac{1}{x-3}) = 2 \cdot \frac{1}{x-3} = \frac{2}{x-3}$

(d) $(g \circ f)(x) = g(f(x)) = g(2x) = \frac{1}{2x-3}$

2. **(a)** $(f \circ g)(0) = f(g(0)) = f(2 - 0^2) = f(2) = 2 \cdot 2 - 3 = 1$

(b) $(g \circ f)(0) = g(f(0)) = g(2 \cdot 0 - 3) = g(-3) = 2 - (-3)^2 = 2 - 9 = -7$

(c) $(f \circ f)(4) = f(f(4)) = f(2 \cdot 4 - 3) = f(5) = 2 \cdot 5 - 3 = 7$

(d) $(g \circ g)(-3) = g(g(-3)) = g(2 - (-3)^2) = g(-7) = 2 - (-3)^2 = 2 - 49 = -47$

(e) $(f \circ g)(-1) = f(g(-1)) = f(2 - (-1)^2) = f(1) = 2 \cdot 1 - 3 = -1$

(f) $(g \circ f)(-3) = g(f(-3)) = g(2 \cdot (-3) - 3) = g(-9) = 2 - (-9)^2 = 2 - 81 = -79$

(g) $(f \circ g)(x) = f(g(x)) = f(2 - x^2) = 2 \cdot (2 - x^2) - 3 = 4 - 2x^2 - 3 = 1 - 2x^2$

(h) $(g \circ f)(x) = g(f(x)) = g(2x - 3) = 2 - (2x - 3)^2$

$$= 2 - (4x^2 - 12x + 9) = 2 - 4x^2 + 12x - 9$$

$$\Rightarrow (g \circ f)(x) = -4x^2 + 12x - 7$$

(i) $(f \circ f)(x) = f(f(x)) = f(2x - 3) = 2(2x - 3) - 3 = 4x - 6 - 3 = 4x - 9$

(j) $(g \circ g)(x) = g(g(x)) = g(2 - x^2) = g(2 - (2 - x^2)^2)$

$$= 2 - (4 - 4x^2 + x^4) = 2 - 4 + 4x^2 - x^4$$

$$\Rightarrow (g \circ g)(x) = -x^4 + 4x^2 - 2$$

3. In each question, when finding the domain of $f \circ g$, check the following two conditions:

- the input x is in the domain of g, because the first rule to be applied to x is g, the inside function of $f(g(x))$.
- the output $g(x)$ is in the domain of f (the range of g must be either equal to or a subset of the domain of f).

The same applies when finding the domain of $g \circ f$: x must be in the domain of f, because the first rule to be applied to x is f, the inside function of $g(f(x))$, and $f(x)$ is in the domain of g.

(a)

	f	g
Domain	$x \in \mathbb{R}$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}$	$y \in \mathbb{R}$

$(f \circ g)(x) = f(g(x)) = f(2+3x) = 4(2+3x) - 1 = 8 + 12x - 1 = 7 + 12x$

$x \in \mathbb{R} \Rightarrow 2 + 3x \in \mathbb{R}$, this set of values is the same as the domain for f, so the domain of $f \circ g$ is also $x \in \mathbb{R}$.

$(g \circ f)(x) = g(f(x)) = g(4x - 1) = 2 + 3(4x - 1) = 2 + 12x - 3 = 12x - 1$

$x \in \mathbb{R} \Rightarrow 4x - 1 \in \mathbb{R}$, this set of values is the same as the domain for g, so the domain of $g \circ f$ is also $x \in \mathbb{R}$.

(b)

	f	g
Domain	$x \in \mathbb{R}$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}$, $y \geq 1$	$y \in \mathbb{R}$

$(f \circ g)(x) = f(g(x)) = f(-2x) = (-2x)^2 + 1 = 4x^2 + 1$

$x \in \mathbb{R} \Rightarrow -2x \in \mathbb{R}$, this set of values is the same set of values as the domain for f, so the domain of $f \circ g$ is also $x \in \mathbb{R}$.

$(g \circ f)(x) = g(f(x)) = g(x^2 + 1) = -2(x^2 + 1) = -2x^2 - 2$

$x \in \mathbb{R} \Rightarrow 1 + x^2 \geq 1$, this set of values is a subset of the domain of g, so the domain of $g \circ f$ is also $x \in \mathbb{R}$.

(c)

	f	g
Domain	$x \in \mathbb{R}$, $x \geq -1$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}$, $y \geq 0$	$y \in \mathbb{R}$, $y \geq 1$

$(f \circ g)(x) = f(g(x)) = f(1 + x^2) = \sqrt{1 + x^2 + 1} = \sqrt{x^2 + 2}$

$x \in \mathbb{R} \Rightarrow 1 + x^2 \geq 1$, this set of values is a subset of the domain of f, so the domain of $f \circ g$ is also $x \in \mathbb{R}$.

$(g \circ f)(x) = g(f(x)) = g(\sqrt{x+1}) = 1 + (\sqrt{x+1})^2 = 1 + x + 1 = x + 2$

$x \geq -1 \Rightarrow \sqrt{x+1} \geq 0$, this set of values is a subset of the domain of g, so the domain of $g \circ f$ is also $x \geq -1$.

(d)

	f	g
Domain	$x \in \mathbb{R}$, $x \neq -4$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}, y \neq 0$	$y \in \mathbb{R}$

$$(f \circ g)(x) = f(g(x)) = f(x-1) = \frac{2}{x-1+4} = \frac{2}{x+3}$$

$x \in \mathbb{R} \Rightarrow x - 1 \in \mathbb{R}$, but this set of values is not a subset of the domain of f. To be able to compose the two functions, the x-value, which is the input of g resulting in an output of $y = -4$, must be excluded from the domain of g.

$x - 1 = -4 \Rightarrow x = -3$, so the domain of $(f \circ g)(x)$ is $x \in \mathbb{R}, x \neq -3$.

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{2}{x+4}\right) = \frac{2}{x+4} - 1 = \frac{2-x-4}{x+4} = \frac{-x-2}{x+4} = -\frac{x+2}{x+4}$$

$x \in \mathbb{R}, x \neq -4 \Rightarrow \frac{2}{x+4} \neq 0$, this set of values is a subset of the domain of g, so the domain of $(g \circ f)(x)$ is also $x \in \mathbb{R}, x \neq -4$.

(e)

	f	g
Domain	$x \in \mathbb{R}$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}$	$y \in \mathbb{R}$

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{x-5}{3}\right) = 3 \cdot \frac{x-5}{3} + 5 = x - 5 + 5 = x$$

$x \in \mathbb{R} \Rightarrow \frac{x-5}{3} \in \mathbb{R}$, this set of values is the same as the domain for f, so the domain of $f \circ g$ is also $x \in \mathbb{R}$.

$$(g \circ f)(x) = g(f(x)) = g(3x+5) = \frac{3x+5-5}{3} = \frac{3x}{3} = 3$$

$x \in \mathbb{R} \Rightarrow 3x - 5 \in \mathbb{R}$, this set of values is the same as the domain for g, so the domain of $g \circ f$ is also $x \in \mathbb{R}$.

(f)

	f	g
Domain	$x \in \mathbb{R}$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}$	$y \leq 1$, $y \in \mathbb{R}$

$$(f \circ g)(x) = f(g(x)) = f(\sqrt[3]{1-x^2}) = 2 - (\sqrt[3]{1-x^2})^3$$
$$= 2 - (1 - x^2) = 2 - 1 + x^2 = x^2 + 1$$

$x \in \mathbb{R} \Rightarrow \sqrt[3]{1-x^2} \leq 1$, this set of values is a subset of the domain of f, so the domain of $f \circ g$ is also $x \in \mathbb{R}$.

$$(g\circ f)(x) = g(f(x)) = g(2-x^3) = \sqrt[3]{1-(2-x^3)^2} = \sqrt[3]{1-(4-4x^3+x^6)}$$
$$= \sqrt[3]{-x^6+4x^3-3}$$

$x \in \mathbb{R} \Rightarrow 2-x^3 \in \mathbb{R}$, this set of values is the same as the domain for g, so the domain of $g\circ f$ is also $x \in \mathbb{R}$.

(g)

	f	g
Domain	$x \in \mathbb{R}$, $x \neq 4$	$x \in \mathbb{R}$, $x \neq 0$
Range	$y \in \mathbb{R}$	$y \in \mathbb{R}$, $y \neq 0$

$$(f\circ g)(x) = f(g(x)) = f\left(\frac{1}{x^2}\right) = \frac{2\frac{1}{x^2}}{4-\frac{1}{x^2}} = \frac{\frac{2}{x^2}}{\frac{4x^2-1}{x^2}} = \frac{2}{4x^2-1}$$

$x \in \mathbb{R}, x \neq 0 \Rightarrow \frac{1}{x^2} > 0$, but this set of values is not a subset of the domain of f. To be able to compose the two functions, the x-value, which is the input of g resulting in an output of $y = 4$, must be excluded from the domain of g.

$\frac{1}{x^2} = 4 \Rightarrow \frac{1}{4} = x^2 \Rightarrow x = \pm\frac{1}{2}$, so the domain of $f\circ g$ is: $x \in \mathbb{R}, x \neq 0, x \neq \pm\frac{1}{2}$.

$$(g\circ f)(x) = g(f(x)) = g\left(\frac{2x}{4-x}\right) = \frac{1}{\left(\frac{2x}{4-x}\right)^2} = \frac{(4-x)^2}{4x^2}$$

$x \in \mathbb{R}, x \neq 4 \Rightarrow \frac{2x}{4-x} \neq 0$, but this is not a subset of the domain of g. To be able to compose the two functions, the x-value, which is the input of f resulting in an output of $y = 4$, must be excluded from the domain of f.

$\frac{2x}{4-x} = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$, so the domain of $g\circ f$ is: $x \in \mathbb{R}, x \neq 4, x \neq 0$.

(h)

	f	g
Domain	$x \in \mathbb{R}$, $x \neq -3$	$x \in \mathbb{R}$, $x \neq -3$
Range	$y \in \mathbb{R}$, $y \neq -3$	$y \in \mathbb{R}$, $y \neq -3$

$$(f\circ g)(x) = f(g(x)) = f\left(\frac{2}{x+3}-3\right) = \frac{2}{\frac{2}{x+3}-3+3} - 3 = 2\cdot\frac{x+3}{2} = xx \in \mathbb{R},$$

$x \neq -3 \Rightarrow \frac{2}{x+3} - 3 \neq -3$, this set of values is the same as the domain for f, so the domain of $f\circ g$ is also $x \in \mathbb{R}, x \neq -3$.

$(g \circ f)(x) = g(f(x)) = g(\frac{2}{x+3} - 3) = x$

$x \in \mathbb{R}, x \neq -3 \Rightarrow \frac{2}{x+3} - 3 \neq -3$, this set of values is the same as the domain for g, so the domain of $g \circ f$ is also $x \in \mathbb{R}, x \neq -3$.

(i)

	f	g
Domain	$x \in \mathbb{R}$, $x \neq 1$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}$, $y \neq 1$	$y \in \mathbb{R}$, $y \geq -1$

$$(f \circ g)(x) = f(g(x)) = f(x^2 - 1) = \frac{x^2 - 1}{x^2 - 1 - 1} = \frac{x^2 - 1}{x^2 - 2}$$

$x \in \mathbb{R} \Rightarrow x^2 - 1 \geq -1$, but this set of values is not a subset of the domain of *f*. To be able to compose the two functions, the *x*-value, which is the input of g resulting in an output of $y = 1$, must be excluded from the domain of g.

$x^2 - 1 = 1 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$, so the domain of $f \circ g$ is $x \in \mathbb{R}, x \neq \pm\sqrt{2}$.

$$(g \circ f)(x) = g\big(f(x)\big) = g\left(\frac{x}{x-1}\right) = \left(\frac{x}{x-1}\right)^2 - 1 = \frac{x^2}{(x-1)^2} - 1$$

$$= \frac{x^2 - (x^2 - 2x + 1)}{(x-1)^2} = \frac{2x - 1}{(x-1)^2}$$

$x \in \mathbb{R}, x \neq 1 \Rightarrow \frac{x}{x-1} \neq 1$, this set of values is a subset of the domain of g, so the domain of $g \circ f$ is also$x \in \mathbb{R}, x \neq 1$.

4. **(a)**

	g	h
Domain	$x \in \mathbb{R}$, $x \geq 1$	$x \in \mathbb{R}$
Range	$y \geq 0$	$y \in \mathbb{R}$, $y \leq 10$

$(g \circ h)(x) = g(h(x)) = g(10 - x^2) = \sqrt{10 - x^2 - 1} = \sqrt{9 - x^2}$

$x \in \mathbb{R} \Rightarrow 10 - x^2 \leq 10$, this is not a subset of the domain of *f*. To be able to compose the two functions, only the *x*-values which are common to both the domain of *g* (real numbers greater than or equal to 1) and the range of *h* (real numbers less than or equal to 10) are acceptable, so only real numbers between 1 and 10, including 1 and 10, can be inputs for *g*. This means that the domain of $g \circ h$ will not contain the *x*-values that will lead to outputs which are less than 1.

$10 - x^2 < 1 \Rightarrow 9 < x^2 \Rightarrow x < -3, x > 3$, so the domain of $g \circ h$ is: $-3 \leq x \leq 3$.

The outputs of $g \circ h$, corresponding to inputs taking values from the set $\{x: -3 \leq x \leq 3\}$, will be elements of the set $\{y: 0 \leq y \leq 3\}$, so the range of $g \circ h$ is $0 \leq y \leq 3$.

(b) $(h \circ g)(x) = h(g(x)) = h(\sqrt{x-1}) = 10 - (\sqrt{x-1})^2 = 10 - x + 1 = -x + 11$

$x \geq 1 \Rightarrow \sqrt{x-1} \geq 0$, this is a subset of the domain of h, so the domain of $h \circ g$ is also $x \geq 1$.

The range of $h \circ g$ is $y \leq 10$, as the restriction on the domain of h does not impact the range of the quadratic $y = 10 - x^2$.

5. **(a)**

	f	g
Domain	$x \in \mathbb{R}$, $x \neq 0$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}$, $y \neq 0$	$y \in \mathbb{R}$, $y \leq 10$

$(f \circ g)(x) = f(g(x)) = f(10 - x^2) = \frac{1}{10-x^2}$

$x \in \mathbb{R} \Rightarrow 10 - x^2 \leq 10$, this set of values is not a subset of the domain of f. To be able to compose the two functions, the x-value, which is the input of g resulting in an output of $y = 0$, must be excluded from the domain of g.

$10 - x^2 = 0 \Rightarrow 10 = x^2 \Rightarrow x = \pm\sqrt{10}$, so the domain of $f \circ g$ is: $x \in \mathbb{R}, x \neq \pm\sqrt{10}$.

The outputs of $f \circ g$ will be elements of the set $\{y: y \in \mathbb{R}, y \neq 10\}$, as the input of g cannot be 0, so the range of $f \circ g$ is: $y \in \mathbb{R}, y \neq 0$.

(b) $(g \circ f)(x) = g(f(x)) = g(\frac{1}{x}) = 10 - \frac{1}{x^2}$

$x \in \mathbb{R}, x \neq 0 \Rightarrow \frac{1}{x} \neq 0$, this set of values is a subset of the domain of g, so the domain of $g \circ f$ is also $x \in \mathbb{R}, x \neq 0$.

The outputs of $g \circ f$, corresponding to inputs taking values from the set $\{x: x \in \mathbb{R}, x \neq 0\}$, will be elements of the set $\{y: y \in \mathbb{R}, y < 10\}$, ($y = 10$ is obtained for $x = 0$, which is not in the domain), so the range of $g \circ f$ is: $y \in \mathbb{R}, y < 10$.

6. In each of the following, try to identify the rules transforming the input x into the expression given by $f(x)$, and take into consideration the order in which the two functions are combined.

(a) $f(x) = (x+3)^2 \Rightarrow 3$ is added to x, and the result is squared
$\Rightarrow h(x) = x + 3, g(x) = x^2$

(b) $f(x) = \sqrt{x-5} \Rightarrow 5$ is subtracted from x, and the result is square rooted
$\Rightarrow h(x) = x - 5, g(x) = \sqrt{x}$

(c) $f(x) = 7 - \sqrt{x} \Rightarrow x$ is square rooted, and the result is subtracted from 7
$\Rightarrow h(x) = \sqrt{x}, g(x) = 7 - x$

(d) $f(x) = \frac{1}{x+3} \Rightarrow 3$ is added to x, and the reciprocal of the result is computed
$\Rightarrow h(x) = x + 3, g(x) = \frac{1}{x}$

(e) $f(x) = 10^{x+1}$ $\Rightarrow$1 is added to x, this result is then the power to which 10 is raised $\Rightarrow h(x) = x + 1, g(x) = 10^x$

(f) $f(x) = \sqrt[3]{x - 9}$ $\Rightarrow$ 9 is subtracted from x, and the result is cube rooted $\Rightarrow h(x) = x - 9, g(x) = \sqrt[3]{x}$

(g) $f(x) = |x^2 - 9|$ $\Rightarrow$ 9 is subtracted from the square of x, and the absolute value of result is taken $\Rightarrow h(x) = x^2 - 9, g(x) = |x|$

(h) $f(x) = \frac{1}{\sqrt{x-5}}$ $\Rightarrow$ the square root of the difference between x and 5, and the reciprocal of the result is computed $\Rightarrow h(x) = \sqrt{x - 5}, g(x) = \frac{1}{x}$

7. **(a)**

	f	g
Domain	$x \in \mathbb{R}$, $x \geq 0$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}$, $y \geq 0$	$y \in \mathbb{R}$, $y \geq 1$

$(f \circ g)(x) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1 + 1} = \sqrt{x^2 + 2}$

$x \in \mathbb{R} \Rightarrow x^2 + 1 \geq 1$, this is a subset of the domain of f, so the domain of $f \circ g$ is also $x \in \mathbb{R}$.

(b)

	f	g
Domain	$x \in \mathbb{R}$, $x \neq 0$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}$, $y \neq 0$	$y \in \mathbb{R}$

$(f \circ g)(x) = f(g(x)) = f(x + 3) = \dfrac{1}{x + 3}$

$x \in \mathbb{R} \Rightarrow x + 3 \in \mathbb{R}$, but this is not a subset of the domain of f. To be able to compose the two functions, the x-value which is the input of g resulting in an output of $y = 0$ must be excluded from the domain of g.

$x + 3 = 0 \Rightarrow x = -3$, so the domain of $f \circ g$ is $x \in \mathbb{R}, x \neq -3$

(c)

	f	g
Domain	$x \in \mathbb{R}$, $x \neq \pm 1$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}$, $y \neq 0$	$y \in \mathbb{R}$

$$(f \circ g)(x) = f(g(x)) = f(x+1) = \frac{3}{(x+1)^2 - 1} = \frac{3}{x^2 + 2x + 1 - 1} = \frac{3}{x^2 + 2x}$$

$x \in \mathbb{R} \Rightarrow x + 1 \in \mathbb{R}$, but this set of values is not a subset of the domain of *f*. To be able to compose the two functions, the *x*-values, which are the inputs of g resulting in the outputs $y = \pm 1$, must be excluded from the domain of *f*.

$x + 1 = \pm 1 \Rightarrow x = -1 \pm 1 \Rightarrow x = 0, x = -2$,
so the domain of $f \circ g$ is $x \in \mathbb{R}, x \neq -2, x \neq 0$

(d)

	f	g
Domain	$x \in \mathbb{R}$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}$	$y \in \mathbb{R}$

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{x}{2}\right) = 2 \cdot \frac{x}{2} + 3 = x + 3$$

$x \in \mathbb{R} \Rightarrow \frac{x}{2} \in \mathbb{R}$, this set of values is the same set of values as the domain for *f*, so the domain of $f \circ g$ is also $x \in \mathbb{R}$.