Continuity and Intermediate Value Theorem

1. **[P]** **True** or **False**: Let $P(t)$ =the cost of parking in New York City's parking garages for t hours. So,

$$P(t) = \$20 \text{ per hour or fraction thereof}$$

For example, if you are in the garage for two hours and one minute, you pay \$60. If t_0 closely approximates some time, T, then $P(t_0)$ closely approximates $P(T)$. Be prepared to justify your answer.

Answer: False. Most students will answer correctly. However, explain how no matter how close t_0 is to T, $P(t_0)$ might not be close to $P(t)$.

2. **[Q]** A drippy faucet adds one milliliter to the volume of water in a tub at precisely one second intervals. Let f be the function that represents the volume of water in the tub at time t.

   (a) f is a continuous function at every time t

   (b) f is continuous for all t other than the precise instants when the water drips into the tub

   (c) f is not continuous at any time t

   (d) not enough information to know where f is continuous.

Answer: (b) Students should be encouraged to draw $f(t)$ and should be able to see the answer quickly. Note that (a) can also be the correct answer, depending on the model that students use for the phenomenon: if the drop of water gradually merges with the water in the tub, the function is continuous with respect to time.

3. **[P]** A drippy faucet adds one milliliter to the volume of water in a tub at precisely one second intervals. Let g be the function that represents the volume of water in the tub as a function of the depth of the water, x, in the tub.

   (a) g is a continuous function at every depth x

   (b) there are some values of x at which g is not continuous

   (c) g is not continuous at any depth, x

   (d) not enough information to know where g is continuous.

Answer: (a) Again, students should be encouraged to draw the graph of $g(x)$. It is interesting to compare this to the previous question. It should be pointed out the difference between the independent variables in the two problems.

4. **[Q]** You know the following statement is true:

If $f(x)$ is a polynomial, then $f(x)$ is continuous.

Which of the following is also true?

(a) If $f(x)$ is not continuous, then it is not a polynomial.

(b) If $f(x)$ is continuous, then it is a polynomial.

(c) If $f(x)$ is not a polynomial, then it is not continuous.

Answer: (a). This may seem like an easy logic question, but students tend to have difficulties; it might be a good time to review some logic. This question prepares students for reasoning that even if differentiability implies continuity, continuity does not imply differentiability. Ask for examples of functions that are continuous, but not polynomials.

5. **[D]** You decide to estimate e^2 by squaring longer decimal approximations of $e = 2.71828\ldots$.

   (a) This is a good idea because e is a rational number.

   (b) This is a good idea because $y = x^2$ is a continuous function.

   (c) This is a bad idea because e is irrational.

   (d) This is a good idea because $y = e^x$ is a continuous function.

   Answer: (b). First recall problem 4 in section 2.1, or present that problem if you have not used it. Students might have problems differentiating between (b) and (d). Point out that we estimate e^2 by considering the limit $\lim_{x \to e} x^2$ as opposed to $\lim_{x \to 2} e^x$.

   Use the following 3 problems in a sequence to illustrate the use of IVT.

6. **[Q]** **True** or **False**. You were once exactly 3 feet tall.

   Answer: True. All students were less than 3 ft tall when they were born, and they should be taller than 3 ft, so by IVT, assuming their growth was continuous, at some point in their life they must have been 3 ft tall. Stress that students may not know WHEN exactly they were 3 feet tall. Instructors should allow the possibility that growth is not continuous, for example some students may reason that they grew by a molecule at a time.

7. **[P]** **True** or **False**. At some time since you were born your weight in pounds equaled your height in inches.

   Answer: True. Students must consider the difference of two functions: $f(t) = \text{height}(t) - \text{weight}(t)$, functions of time. At birth, $f(\text{birth}) > 0$ and right now, $f(\text{now}) < 0$, hence by IVT $f(T) = 0$, where T is some time in the past. It is important to stress that this technique of looking at the difference of two functions is recurrent in calculus.

8. **[D]** **True** or **False**. Along the Equator, there are two diametrically opposite sites that have exactly the same temperature at the same time.

   Answer: True. Do not use this problem unless you have already worked through the previous two problems. This problem is not at all intuitive. Again, students must consider a difference: the difference in temperature between the diametrically opposite sites.

9. **[Q]** Suppose that during half-time at a basketball game the score of the home team was 36 points.

   True or **False**: There had to be at least one moment in the first half when the home team had exactly 25 points.

   Answer: False. Scoring in basketball is not continuous, so the IVT does not apply here. Students will probably enjoy thinking about this problem.

10. **[D]** **True** or **False**. $x^{100} - 9x^2 + 1$ has a root in $[0, 2]$.

    Answer: True. This problem is not a direct application of IVT, plugging in 0 and 2, we get positive numbers, so the student must choose some other number in $[0, 2]$ to test. Choosing 0 and 1, or 1 and 2, IVT immediately applies.

11. **[P]** **True** or **False**. The function $f(x)$ defined below is continuous at $x = 0$.

$$f(x) = \begin{cases} x^2 & x \text{ is rational} \\ -x^2 & x \text{ is irrational} \end{cases}$$

    Answer: True. This problem should not be that hard if students have seen this function before.