La derivada, el creixement

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Índex

Aquest és un document de treball INTERN, en fase de discussió i molt preliminar. No en feu difussió, sisplau.

(Enllaç al document principal)
(Enllaç a la versió pdf d'aquest document)
(Enllaç a la font LaTeX)
(Enllaç als fitxers de les figures)

1 Objectius d'aprenentatge

- Definició i interpretació de la derivada

- La derivada i la tangent
- L'aproximació lineal
  - El mètode de Newton
- Repàs de les regles de càlcul de derivades
- xx

2 Prerequisits

El càlcul de derivades de funcions polinòmiques, potencials i racionals. Se'ls poden donar referències, exemples i eines (geogebra, WolframAlpha) per ajudar a repassar.

Per exemple, exercicis del IB-An&Appr 13-4:

Exercise 13.4

1. Find the derivative of each function.

(a) $y = (3x - 8)^4$ **(b)** $y = \sqrt{1 - x}$

(c) $y = \frac{3}{x^3}$ **(d)** $y = \frac{x^3 + 1}{2x}$

(e) $y = (x^2 + 4)^{-2}$ **(f)** $y = \frac{x}{x + 1}$

(g) $y = \frac{1}{\sqrt{x + 2}}$ **(h)** $y = (2x^2 - 1)^3$

(i) $y = x\sqrt{1 - x}$ **(j)** $y = \frac{1}{3x^2 - 5x + 7}$

(k) $y = \sqrt[3]{2x + 5}$ **(l)** $y = (2x - 1)^3(x^4 + 1)$

(m) $y = \sqrt{3x^2 - 2}$ **(n)** $y = \frac{x^2}{x + 2}$

(o) $y = \frac{x + 1}{x - 1}$

2. Given that $y = \frac{1}{x^4}$, find:

(a) $\frac{dy}{dx}$ **(b)** $\frac{d^2y}{dx^2}$

3. A curve has equation $y = (x - 1)(2x + 1)^2$

(a) Find $\frac{dy}{dx}$ by first expanding the right side of the equation and applying the power rule.

(b) Find $\frac{dy}{dx}$ by applying the product rule and chain rule.

4. Consider the function $f(x) = \frac{x^2 - 3x + 4}{(x + 1)^2}$

(a) Show that $f'(x) = \frac{5x - 11}{(x + 1)^3}$

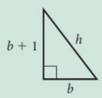
(b) Show that $f''(x) = \frac{-10x + 38}{(x + 1)^4}$

5. Find the first and second derivatives of the function $f(x) = \frac{x - a}{x + a}$, $a \in \mathbb{R}$

6. Given that $y = x\sqrt{x + 1}$, show that $\frac{dy}{dx} = \frac{3x + 2}{2\sqrt{x + 1}}$

7. If $y = x^4 - 6x^2$, show that y, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ are all negative on the interval $0 < x < 1$, but that $\frac{d^3y}{dx^3}$ is positive on the same interval.

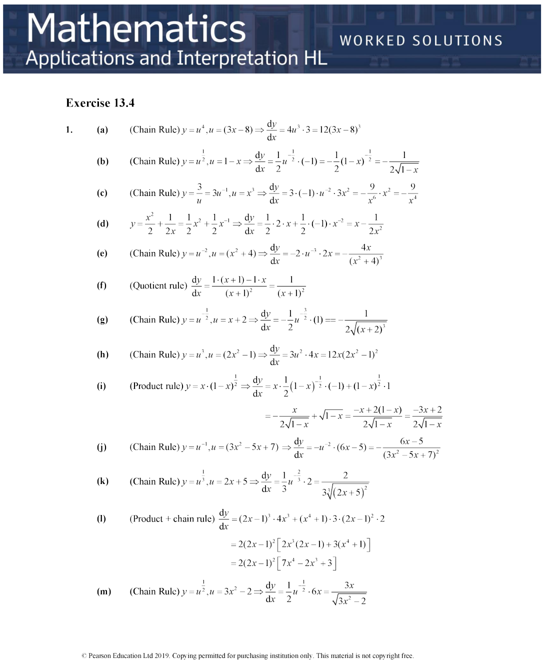
8. Consider the right-angled triangle with sides b and $b + 1$ and hypotenuse h.



(a) Express h as a function of b.

(b) Show that $\frac{dh}{db} = \frac{2b + 1}{\sqrt{2b^2 + 2b + 1}}$

amb les seves solucions



Mathematics
Applications and Interpretation HL
WORKED SOLUTIONS

Exercise 13.4

1. **(a)** (Chain Rule) $y = u^4, u = (3x-8) \Rightarrow \frac{dy}{dx} = 4u^3 \cdot 3 = 12(3x-8)^3$

(b) (Chain Rule) $y = u^{\frac{1}{2}}, u = 1-x \Rightarrow \frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \cdot (-1) = -\frac{1}{2}(1-x)^{-\frac{1}{2}} = -\frac{1}{2\sqrt{1-x}}$

(c) (Chain Rule) $y = \frac{3}{u} = 3u^{-1}, u = x^3 \Rightarrow \frac{dy}{dx} = 3 \cdot (-1) \cdot u^{-2} \cdot 3x^2 = -\frac{9}{x^6} \cdot x^2 = -\frac{9}{x^4}$

(d) $y = \frac{x^2}{2} + \frac{1}{2x} = \frac{1}{2}x^2 + \frac{1}{2}x^{-1} \Rightarrow \frac{dy}{dx} = \frac{1}{2} \cdot 2 \cdot x + \frac{1}{2} \cdot (-1) \cdot x^{-2} = x - \frac{1}{2x^2}$

(e) (Chain Rule) $y = u^{-2}, u = (x^2+4) \Rightarrow \frac{dy}{dx} = -2 \cdot u^{-3} \cdot 2x = -\frac{4x}{(x^2+4)^3}$

(f) (Quotient rule) $\frac{dy}{dx} = \frac{1 \cdot (x+1) - 1 \cdot x}{(x+1)^2} = \frac{1}{(x+1)^2}$

(g) (Chain Rule) $y = u^{-\frac{1}{2}}, u = x+2 \Rightarrow \frac{dy}{dx} = -\frac{1}{2}u^{-\frac{3}{2}} \cdot (1) == -\frac{1}{2\sqrt{(x+2)^3}}$

(h) (Chain Rule) $y = u^3, u = (2x^2-1) \Rightarrow \frac{dy}{dx} = 3u^2 \cdot 4x = 12x(2x^2-1)^2$

(i) (Product rule) $y = x \cdot (1-x)^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = x \cdot \frac{1}{2}(1-x)^{-\frac{1}{2}} \cdot (-1) + (1-x)^{\frac{1}{2}} \cdot 1$

$$= -\frac{x}{2\sqrt{1-x}} + \sqrt{1-x} = \frac{-x+2(1-x)}{2\sqrt{1-x}} = \frac{-3x+2}{2\sqrt{1-x}}$$

(j) (Chain Rule) $y = u^{-1}, u = (3x^2-5x+7) \Rightarrow \frac{dy}{dx} = -u^{-2} \cdot (6x-5) = -\frac{6x-5}{(3x^2-5x+7)^2}$

(k) (Chain Rule) $y = u^{\frac{1}{3}}, u = 2x+5 \Rightarrow \frac{dy}{dx} = \frac{1}{3}u^{-\frac{2}{3}} \cdot 2 = \frac{2}{3\sqrt[3]{(2x+5)^2}}$

(l) (Product + chain rule) $\frac{dy}{dx} = (2x-1)^3 \cdot 4x^3 + (x^4+1) \cdot 3 \cdot (2x-1)^2 \cdot 2$

$$= 2(2x-1)^2\left[2x^3(2x-1) + 3(x^4+1)\right]$$
$$= 2(2x-1)^2\left[7x^4 - 2x^3 + 3\right]$$

(m) (Chain Rule) $y = u^{\frac{1}{2}}, u = 3x^2-2 \Rightarrow \frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \cdot 6x = \frac{3x}{\sqrt{3x^2-2}}$

3 Guia pel professor

Cal posar l'èmfasi en els conceptes, la idea de mesurar el canvi, el creixement. Els exemples d'aplicació a l'economia i el cas de les epidèmies han de ser referents clau.

3.1 Presentació del tema

Donant per suposat que abans de la classe s'han mirat els materials proposats i han fet les activitats autònomes, es proposaran preguntes i *clickers* per provocar la reflexió i discussió en petits grups dels conceptes principals:

- El quocient d'increments
- La derivada com límit del quocient d'increments
- La derivada com a mesura de la variació
- El signe de la derivada com a indicador de creixement/decreixement

3.1.1 Exemples de Clicker

Si tenim $y = f(x)$ i considerem l'interval $[x_1, x_2]$, el quocient d'increments $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ es sol anomenar de diferents maneres. Digues quines són correctes:
QUIZZ
= la variació de y entre x_1 i x_2
= la variació instantània de y en $x = x_1$
== la variació mitjana de y entre x_1 i x_2.
== la ratio de canvi de y respecte x en l'interval $[x_1, x_2]$
== el quocient incremental de y respecte x en l'interval $[x_1, x_2]$
= la raó de canvis de y respecte x en el punt x_1
== la raó de canvis de y respecte x en l'interval $[x_1, x_2]$

Discutim ara aquest exercici:

5. Consider the graph of *f*.

(a) Between which two consecutive points is the average rate of change of the function greatest?

(b) At what points is the instantaneous rate of change of *f*

(i) positive **(ii)** negative **(iii)** zero?

(c) For which two pairs of points is the average rate of change approximately equal?

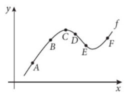


Figure 13.16 Graph for question 5

Solucions:

5. **(a)** The rate of change is greater when the function is steeper, so the average rate of change is greatest between A and B.

(b) **(i)** positive rate of change means that the curve is increasing, so at points A, B and F

(ii) negative rate of change means that the curve is decreasing, so at points D and E

(iii) At point C, where the tangent to the curve is horizontal and the curve is neither Increasing or decreasing.

(c) By eye you can see that the lines BD and EF have similar gradients and so the average rate of change between these two pairs of points are approximately equal.

Discutim quina és la resposta correcta i en cas d'incorrecta explica perquè i dona un exemple que ho mostri.

QUIZZ

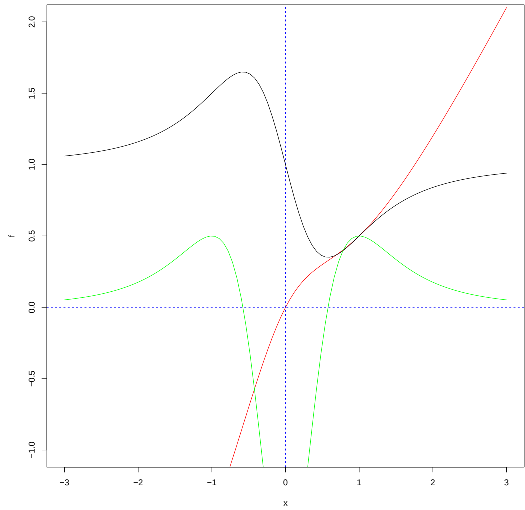
Tenim $y = f(x)$. Considerem l'interval $[x_1, x_2]$ i el quocient d'increments $Q = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ == Si $Q > 0$, podem assegurar que $(f(x_2) > f(x_1)$

= Si el quocient Q és negatiu, la funció és decreixent

= Si $Q > 0$, podem assegurar que el pendent de la gràfica en $x = x_1$ és positiu.

== Si $Q < 0$ el segment que uneix els punts de la gràfica "fa baixada" = Si $Q = 0$, la gràfica és horitzontal.

QUIZZ



En el gràfic adjunt hi ha una funció i les seves dues primeres derivades, identifica-les.

A la col·lecció de preguntes de GoodQuestions Project de Cornell University, especialment a partir de la pàgina 15, hi ha moltes preguntes interessants per plantejar a classe per a la discusió.

Per exemple:

1. **[Q] True** or **False**. If $f(x) = x^{1/3}$ then $f'(0)$ exists.

   Answer: False. $f'(0)$ equals the slope of the tangent at $(0, 0)$.

2. **[P] True** or **False**. If $f(x) = x^{1/3}$ then there is a tangent line at $(0, 0)$.

   Answer: True. This gets students to think about tangent lines and derivatives. A vertical tangent line exists, although the derivative does not.

3. **[P] True** or **False**. The function $f(x) = |x|$ has a derivative at $x = 0$.

   Answer: False. The limit $\lim_{x \to 0} \dfrac{f(x) - 0}{x - 0}$ does not exist, because it equals -1 from the left, and 1 from the right. Thus $f'(0)$ does not exist.

4. **[Q] True** or **False**. The function $g(x) = x|x|$ has a derivative at $x = 1$.

   Answer: True. Easy application of the limit definition of derivative. Students should note that close to 1, $|x| = x$. $g'(1) = 2$.

3.2 Materials bàsics

Capítols 6 i 10 del Pemberton&Rau.

4 Activitats autònomes

Aquí recollim exemples de possibles activitats, per suposat que caldrà seleccionar-ne algunes i complementar-les amb d'altres ja disponibles.

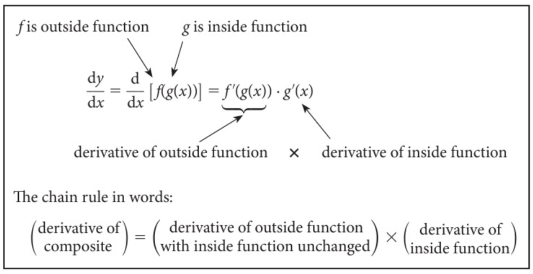
4.0.1 Repassem la regla de la cadena

Ja hem vist casos en què la variació d'una quantitat en funció d'una altra es pot descomposar a través de quantitats intermèdies. Per exemple, la variació del preu del petroli en dòlars segons el temps t en mesos pot ser $P_d(t) = 20+0.4t$ durant un periode determinat. Però el valor de l'euro en funció del dòlar (que considerem constant en el temps) també és una funció que es pot expressar $E(d) = 1.2d$ i per tant el preu del petroli en euros serà $P_e(t) = E(P_d(t))$. La variació del preu del petroli en euros serà

$$\frac{dE}{dt} = \frac{dE}{dD}\frac{dD}{dt}, \qquad \text{o bé} \qquad P_e'(t) = P_e'(P_d(t)) \cdot P_d'(t)$$

Això es pot explicar dient que si el preu en dòlars varia a raó de 0.4 dòlars/mes, i l'euro varia a raó de 1.2 euros/dòlar, la variació del petroli en euros serà de 0.4×1.2 euros per mes.

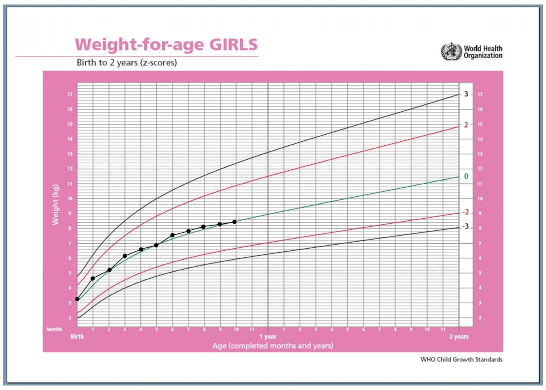
Com a regla general diem:



O bé: *La derivada de la composta és*
la derivada de la de fora (sense canviar la de dins)
multiplicada per
la derivada de la de dintre.

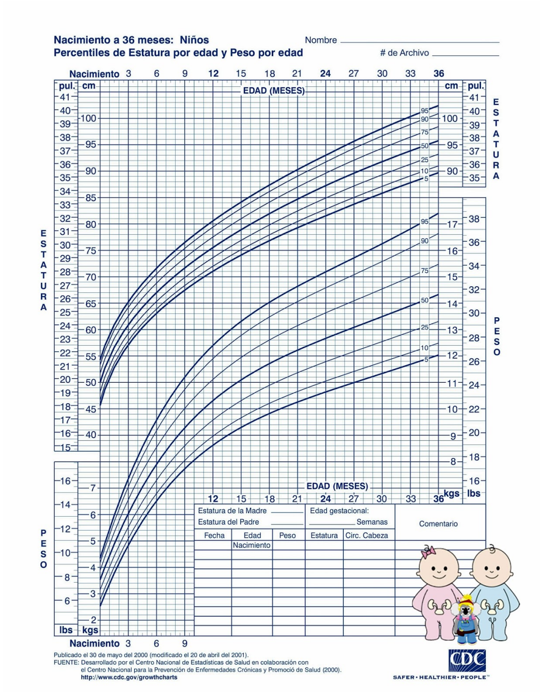
QUIZZ
El preu de la llimonada depén de la demanda $P(D)$ però la demanda depén la temperatura $D = D(T)$. Llavors, la variació del preu en funció de la temperatura serà:
$== P'(D) \cdot D'(T)$
$= P'(D'(T))$
$= P'(D) + D'(T)$

4.0.2 Creixement del pes d'un bebè



En el gràfic veiem el pes d'un bebè al llarg dels primers mesos de vida.

En el mes 5 pesa 6.9 kg, i al mes 7 pesa 7,8 kg. En dos mesos ha augmentat 0,9 kg, el ritme de creixement en aquest periode és de 0,45 kg/mes. Per calcular-ho, dividim l'incement de pes (7,8-6,9=0,9) per l'increment del temps (7-5=2 mesos).



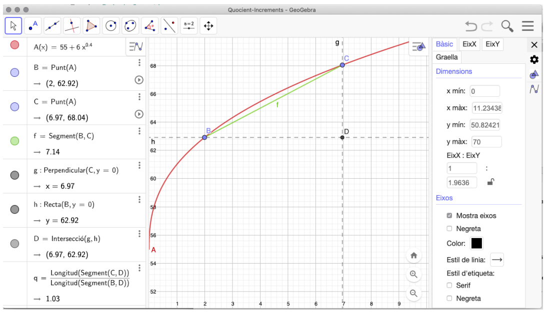
Aqui tenim també la gràfica de les alçades dels bebès segons els mesos d'edad. De les diferents corbes corresponents als diferent quantils (no entrem ara en aquest tema) ens quedem amb la línia del mig, més gruixuda, que correspon al bebè mitjà o promig.

Segons la taula, als 6 mesos el bebè tipus amida 67 cm, i als 9 mesos 74 cm. El seu creixement ha sigut de (74-67)/(9-6)=7/3, aproximadament 2,3 cm/mes. Altre cop calculem el quocient d'increments per calcular el ritme o raó decreixement.

Però el creixement no es produeix de cop, cada mes, sino que es produeix d'una manera continua. Això és el que reflecteix la corba suau que veiem a la figura anterior. Llavors, per amidar el ritme de creixement en un moment determinat, fem el quocient d'increments però el fem en interval de temps molt petits. Per exemple, si volem saber el ritme de creixement en el moment $t = 5$ mesos, prenem dos punts de la corba $t = 5$ i $t = 5.1$ i mirem com ha crescut el bebè en aquest curt periode, i calculem el quocient d'increments:

El ritme de creixment en $t_0 = 5$ és $\dfrac{A(5.1) - A(5)}{5.1 - 5}$

El càlcul el podem veure més clar si prenem una funció senzilla $A(t) = 55 + 4t^{0.2}$ i la grafiquem amb Geogebra. Podeu veure un exemple a https://www.geogebra.org/classic/q8wnqmbb:



etc...

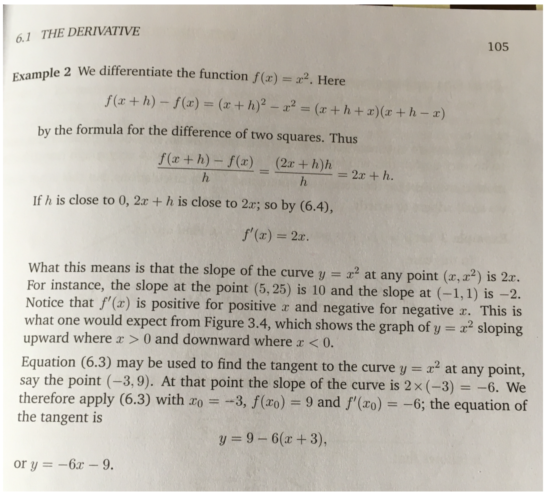
En R podem fer

```
> f<-function(x) 55+ 6*x^0.4
 plot.function(f,0,12)
 > quocIncr <- function(f, x0, h){(f(x0+h)-f(x0))/h}
> quocIncr(f,3,0.1)
[1] 1.229278
> quocIncr(f,2,0.1)
[1] 1.560272
> quocIncr(f,2,1)
[1] 1.394026
> quocIncr(f,2,0.01)
[1] 1.581041
> quocIncr(f,2,0.0001)
[1] 1.583386
```

4.0.3 La derivada és el límit del quocient incremental

Mira't aquests dos exemples de càlcul de la derivada utilitzant la definició:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



6.1 THE DERIVATIVE 105

Example 2 We differentiate the function $f(x) = x^2$. Here

$$f(x+h) - f(x) = (x+h)^2 - x^2 = (x+h+x)(x+h-x)$$

by the formula for the difference of two squares. Thus

$$\frac{f(x+h) - f(x)}{h} = \frac{(2x+h)h}{h} = 2x + h.$$

If h is close to 0, $2x + h$ is close to $2x$; so by (6.4),

$$f'(x) = 2x.$$

What this means is that the slope of the curve $y = x^2$ at any point (x, x^2) is $2x$. For instance, the slope at the point $(5, 25)$ is 10 and the slope at $(-1, 1)$ is -2. Notice that $f'(x)$ is positive for positive x and negative for negative x. This is what one would expect from Figure 3.4, which shows the graph of $y = x^2$ sloping upward where $x > 0$ and downward where $x < 0$.

Equation (6.3) may be used to find the tangent to the curve $y = x^2$ at any point, say the point $(-3, 9)$. At that point the slope of the curve is $2 \times (-3) = -6$. We therefore apply (6.3) with $x_0 = -3$, $f(x_0) = 9$ and $f'(x_0) = -6$; the equation of the tangent is

$$y = 9 - 6(x+3),$$

or $y = -6x - 9$.

we shall adhere to strictly.

Example 3 Let $y = 1/x$; it is assumed that $x > 0$. Find dy/dx.

In this case

$$\Delta y = \frac{1}{x+\Delta x} - \frac{1}{x} = \frac{x-(x+\Delta x)}{x(x+\Delta x)} = -\frac{\Delta x}{x(x+\Delta x)},$$

so

$$\frac{\Delta y}{\Delta x} = -\frac{1}{x(x+\Delta x)}.$$

If $\Delta x \approx 0$, the denominator on the right-hand side is close to x^2, so

$$\frac{\Delta y}{\Delta x} \approx -\frac{1}{x^2}.$$

It follows that

$$\frac{dy}{dx} = -\frac{1}{x^2}.$$

Notice that in this case $dy/dx < 0$. The geometry of this is that since $1/x$ decreases as x increases, the graph of $y = 1/x$ must be downward-sloping. Since $1/x = x^{-1}$, the function $y = 1/x$ is an example of a power function with a negative power, and such functions have already been graphed in Figure 4.4. The result just obtained may be phrased as follows in terms of negative powers:

$$\text{if } y = x^{-1} \text{ then } \frac{dy}{dx} = -x^{-2}.$$

Of course, the result may also be expressed in our original notation for derivatives:

$$\text{if } f(x) = x^{-1} \text{ then } f'(x) = -x^{-2}.$$

Ara repeteix el que has vist per demostrar que si $f(x) = 2 - \frac{1}{3}x^2$, $f'(x) = -\frac{2}{3}$.

O fes aquest 6.1.1. de PemRau:

Exercises

6.1.1 Let $f(x) = 5 - \frac{1}{2}x^2$. Using an algebraic method similar to that of Example 2, find $f'(x)$. Evaluate the slope of the curve $y = f(x)$

(a) where $x = 4$, (b) where $x = -5$.

Find the equation of the tangent to the curve at the point $(2, 3)$.

6.1.1

$$\frac{f(x+h)-f(x)}{h} = \frac{x^2-(x+h)^2}{2h} = -x - \tfrac{1}{2}h,$$

which is close to $-x$ if $|h|$ is small. Hence $f'(x) = -x$.

(a) -4, (b) 5.

$y = 7 - 2x$.

Solució:

Un exercici típic d'examen: Demostra fent servir la definició de la derivada com a límit del quocient d'increments, que per $f(x) = a - bx^2$ on a i b són constants, es té $f'(x) = -2b$.

4.0.4 La derivada com a aproximació lineal

Com que $f'(x_0) \approx \frac{f(x_0+h)-f(x_0)}{h}$ quan h és petit (en valor absolut), podem escriure $f(x_0+h) - f(x_0) \approx |h| f'(x_0)$ quan $|h|$ és petit. Una altra manera d'escriure-ho: Si $y = f(x)$, $\Delta y = y'\Delta x$.

(Repassa el capítol 6.2 de Pemb-Rau is no tens això gaire clar).

QUIZZ

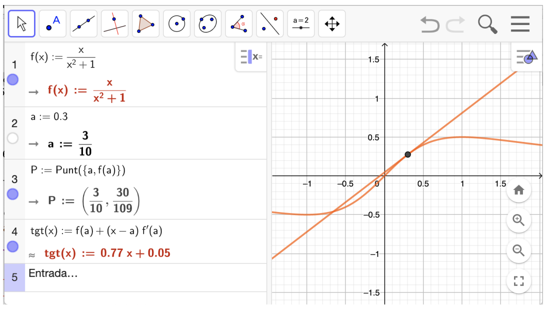
Si el preu depén de la quantitat fabricada segons $P = 2 + 5Q^{1/3}$, calcula el preu marginal per $Q = 3$ i digues quan augmenta el preu aproximadament si la quantitat passa de 3 a 3.01

== El preu augmentarà en 0.0080

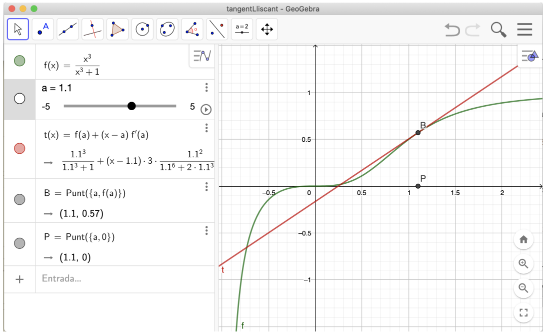
= El preu augmentarà en 0.8012 [No, aixó és la derivada P' per $Q = 3$]

4.0.5 La derivada és el pendent de la recta tangent

Aquí pots veure com dibuixar la tangent a la gràfica d'una funció amb Geogebra.



Fes-ho tu ara per la funció $f(x) = \frac{2x^2}{x^2+1}$ i el punt $a = 1.5$. Porta el teu exemple al seminari.

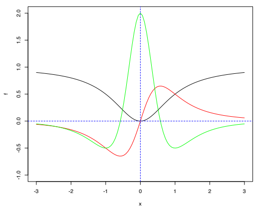
Si fas una construcció com aquesta: 

podràs veure com canvia la tangent quan varia el punt x sobre el que es construeix la tangent.

4.0.6 Funció derivada amb R

R no sap fer derivades simbòlicament, però és molt potent numèricament. Podem demanar-li coses així:

```
# calcul de la funcio derivada numericament

f <- function(x) x^2/(x^2+1)

h <- 0.001

Der <- function(ff){
  return(function(x) (ff(x+h)-ff(x))/h)
}

plot.function(f,-3,3, ylim=c(-1,2))
abline(h=0, col="blue", lty=2)
abline(v=0, col="blue", lty=2)

plot.function(Der(f),-3,3, add=TRUE, col="red")
plot.function(Der(Der(f)),-3,3, add=TRUE, col="
    green")
```

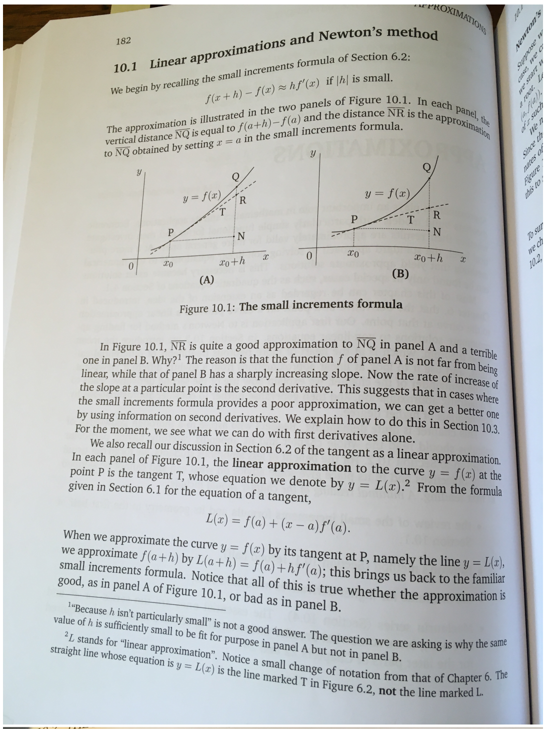


Cal fer això amb precaució: enlloc de la derivada estem calculant el quocient incremental per un increment petit, però "petit" no és una especificació sempre vàlida, segons quina sigui la funció i el punt on es calcula pot haver-hi errors no menyspreables.

4.0.7 El mètode de Newton per resoldre equacions

En el tema 6. Teoremes de Bolzano, valors extrems vem veure com obtenir solucions a equacions amb el mètode de bisecció. Un mètode molt més eficient es pot trobar si enlloc de passar cada vegada d'un interval a la meitat s'aprofita el fet que la derivda ens dona una bona aproximació de la funció.

Llegeix aquestes pàgines del llibre de text: Pem-Rau.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10.1 Linear approximations and Newton's method

We begin by recalling the small increments formula of Section 6.2:

$$f(x+h) - f(x) \approx hf'(x) \quad \text{if } |h| \text{ is small.}$$

The approximation is illustrated in the two panels of Figure 10.1. In each panel, the vertical distance $\overline{NQ}$ is equal to $f(a+h)-f(a)$ and the distance $\overline{NR}$ is the approximation to $\overline{NQ}$ obtained by setting $x = a$ in the small increments formula.

Figure 10.1: **The small increments formula**

In Figure 10.1, $\overline{NR}$ is quite a good approximation to $\overline{NQ}$ in panel A and a terrible one in panel B. Why?[1] The reason is that the function f of panel A is not far from being linear, while that of panel B has a sharply increasing slope. Now the rate of increase of the slope at a particular point is the second derivative. This suggests that in cases where the small increments formula provides a poor approximation, we can get a better one by using information on second derivatives. We explain how to do this in Section 10.3. For the moment, we see what we can do with first derivatives alone.

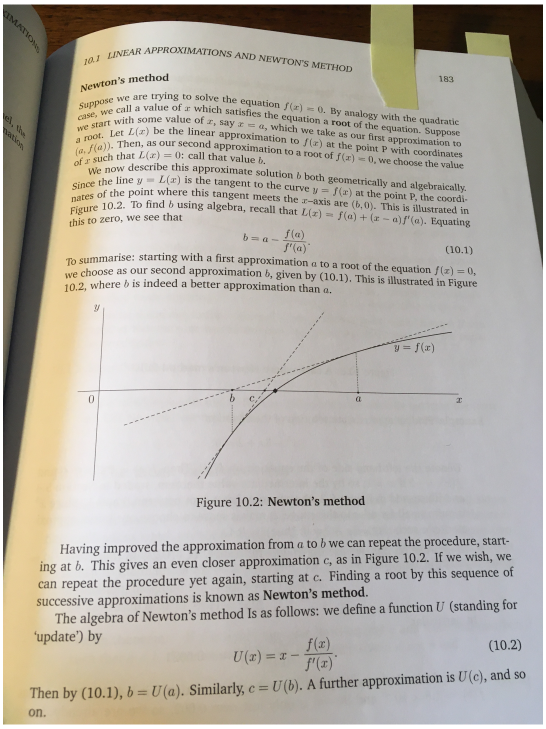
We also recall our discussion in Section 6.2 of the tangent as a linear approximation. In each panel of Figure 10.1, the **linear approximation** to the curve $y = f(x)$ at the point P is the tangent T, whose equation we denote by $y = L(x)$.[2] From the formula given in Section 6.1 for the equation of a tangent,

$$L(x) = f(a) + (x-a)f'(a).$$

When we approximate the curve $y = f(x)$ by its tangent at P, namely the line $y = L(x)$, we approximate $f(a+h)$ by $L(a+h) = f(a)+hf'(a)$; this brings us back to the familiar small increments formula. Notice that all of this is true whether the approximation is good, as in panel A of Figure 10.1, or bad as in panel B.

[1] "Because h isn't particularly small" is not a good answer. The question we are asking is why the same value of h is sufficiently small to be fit for purpose in panel A but not in panel B.

[2] L stands for "linear approximation". Notice a small change of notation from that of Chapter 6. The straight line whose equation is $y = L(x)$ is the line marked T in Figure 6.2, **not** the line marked L.


10.1 LINEAR APPROXIMATIONS AND NEWTON'S METHOD 183

Newton's method

Suppose we are trying to solve the equation $f(x) = 0$. By analogy with the quadratic case, we call a value of x which satisfies the equation a **root** of the equation. Suppose we start with some value of x, say $x = a$, which we take as our first approximation to a root. Let $L(x)$ be the linear approximation to $f(x)$ at the point P with coordinates $(a, f(a))$. Then, as our second approximation to a root of $f(x) = 0$, we choose the value of x such that $L(x) = 0$: call that value b.

We now describe this approximate solution b both geometrically and algebraically. Since the line $y = L(x)$ is the tangent to the curve $y = f(x)$ at the point P, the coordinates of the point where this tangent meets the x-axis are $(b, 0)$. This is illustrated in Figure 10.2. To find b using algebra, recall that $L(x) = f(a) + (x-a)f'(a)$. Equating this to zero, we see that

$$b = a - \frac{f(a)}{f'(a)}. \tag{10.1}$$

To summarise: starting with a first approximation a to a root of the equation $f(x) = 0$, we choose as our second approximation b, given by (10.1). This is illustrated in Figure 10.2, where b is indeed a better approximation than a.

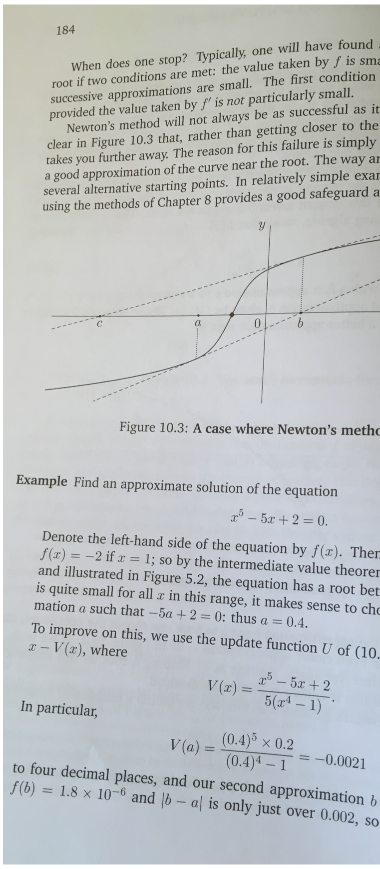
Figure 10.2: **Newton's method**

Having improved the approximation from a to b we can repeat the procedure, starting at b. This gives an even closer approximation c, as in Figure 10.2. If we wish, we can repeat the procedure yet again, starting at c. Finding a root by this sequence of successive approximations is known as **Newton's method**.

The algebra of Newton's method Is as follows: we define a function U (standing for 'update') by

$$U(x) = x - \frac{f(x)}{f'(x)}. \tag{10.2}$$

Then by (10.1), $b = U(a)$. Similarly, $c = U(b)$. A further approximation is $U(c)$, and so on.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When does one stop? Typically, one will have found root if two conditions are met: the value taken by f is sma successive approximations are small. The first condition provided the value taken by f' is *not* particularly small.

Newton's method will not always be as successful as it clear in Figure 10.3 that, rather than getting closer to the takes you further away. The reason for this failure is simply a good approximation of the curve near the root. The way ar several alternative starting points. In relatively simple exar using the methods of Chapter 8 provides a good safeguard a

Figure 10.3: **A case where Newton's metho**

Example Find an approximate solution of the equation

$$x^5 - 5x + 2 = 0.$$

Denote the left-hand side of the equation by $f(x)$. Then $f(x) = -2$ if $x = 1$; so by the intermediate value theorer and illustrated in Figure 5.2, the equation has a root bet is quite small for all x in this range, it makes sense to cho mation a such that $-5a + 2 = 0$: thus $a = 0.4$.

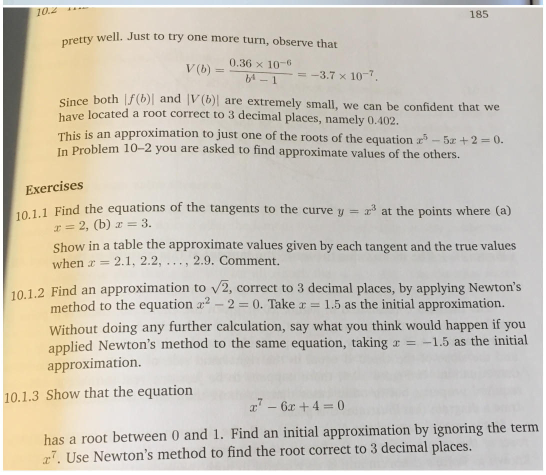
To improve on this, we use the update function U of (10. $x - V(x)$, where

$$V(x) = \frac{x^5 - 5x + 2}{5(x^4 - 1)}.$$

In particular,

$$V(a) = \frac{(0.4)^5 \times 0.2}{(0.4)^4 - 1} = -0.0021$$

to four decimal places, and our second approximation b $f(b) = 1.8 \times 10^{-6}$ and $|b - a|$ is only just over 0.002, so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pretty well. Just to try one more turn, observe that

$$V(b) = \frac{0.36 \times 10^{-6}}{b^4 - 1} = -3.7 \times 10^{-7}.$$

Since both $|f(b)|$ and $|V(b)|$ are extremely small, we can be confident that we have located a root correct to 3 decimal places, namely 0.402.

This is an approximation to just one of the roots of the equation $x^5 - 5x + 2 = 0$. In Problem 10–2 you are asked to find approximate values of the others.

Exercises

10.1.1 Find the equations of the tangents to the curve $y = x^3$ at the points where (a) $x = 2$, (b) $x = 3$.

Show in a table the approximate values given by each tangent and the true values when $x = 2.1, 2.2, \ldots, 2.9$. Comment.

10.1.2 Find an approximation to $\sqrt{2}$, correct to 3 decimal places, by applying Newton's method to the equation $x^2 - 2 = 0$. Take $x = 1.5$ as the initial approximation.

Without doing any further calculation, say what you think would happen if you applied Newton's method to the same equation, taking $x = -1.5$ as the initial approximation.

10.1.3 Show that the equation

$$x^7 - 6x + 4 = 0$$

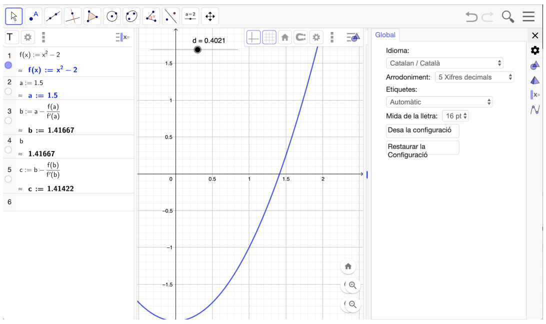
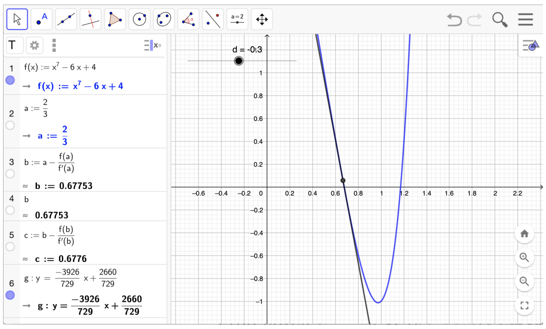
has a root between 0 and 1. Find an initial approximation by ignoring the term x^7. Use Newton's method to find the root correct to 3 decimal places.

. . .

QUIZZ

El mètode de Newton...

Amb Geogebra podem veure fàcilment com treballa el mètode de Newton:

5 Llista d'exercicis

5.0.1 Exercici

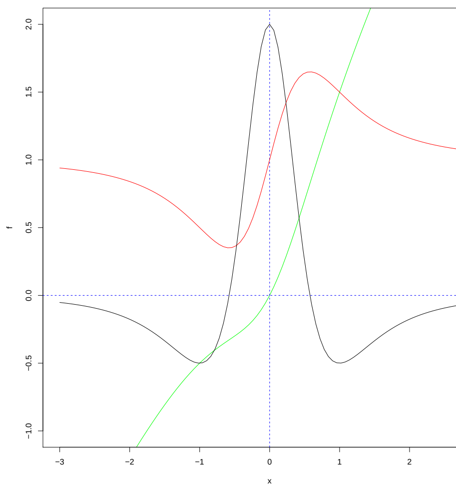
Un exercici típic d'examen: Demostra fent servir la definició de la derivada com a límit del quocient d'increments, que per $f(x) = a + bx^2$ on a i b són constants, es té $f'(x) = 2b$.

6 Suplements avançats

7 Exercicis per exàmens

1. Un exercici típic d'examen: Demostra fent servir la definició de la derivada com a límit del quocient d'increments, que per $f(x) = a + bx^2$ on a i b són constants, es té $f'(x) = 2b$.

2. **QUIZZ**



En el gràfic adjunt hi ha una funció i les seves dues primeres derivades, identifica-les.