Exercise 13.4

1. **(a)** (Chain Rule) $y = u^4, u = (3x-8) \Rightarrow \frac{dy}{dx} = 4u^3 \cdot 3 = 12(3x-8)^3$

(b) (Chain Rule) $y = u^{\frac{1}{2}}, u = 1-x \Rightarrow \frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \cdot (-1) = -\frac{1}{2}(1-x)^{-\frac{1}{2}} = -\frac{1}{2\sqrt{1-x}}$

(c) (Chain Rule) $y = \frac{3}{u} = 3u^{-1}, u = x^3 \Rightarrow \frac{dy}{dx} = 3 \cdot (-1) \cdot u^{-2} \cdot 3x^2 = -\frac{9}{x^6} \cdot x^2 = -\frac{9}{x^4}$

(d) $y = \frac{x^2}{2} + \frac{1}{2x} = \frac{1}{2}x^2 + \frac{1}{2}x^{-1} \Rightarrow \frac{dy}{dx} = \frac{1}{2} \cdot 2 \cdot x + \frac{1}{2} \cdot (-1) \cdot x^{-2} = x - \frac{1}{2x^2}$

(e) (Chain Rule) $y = u^{-2}, u = (x^2+4) \Rightarrow \frac{dy}{dx} = -2 \cdot u^{-3} \cdot 2x = -\frac{4x}{(x^2+4)^3}$

(f) (Quotient rule) $\frac{dy}{dx} = \frac{1 \cdot (x+1) - 1 \cdot x}{(x+1)^2} = \frac{1}{(x+1)^2}$

(g) (Chain Rule) $y = u^{-\frac{1}{2}}, u = x+2 \Rightarrow \frac{dy}{dx} = -\frac{1}{2}u^{-\frac{3}{2}} \cdot (1) == -\frac{1}{2\sqrt{(x+2)^3}}$

(h) (Chain Rule) $y = u^3, u = (2x^2-1) \Rightarrow \frac{dy}{dx} = 3u^2 \cdot 4x = 12x(2x^2-1)^2$

(i) (Product rule) $y = x \cdot (1-x)^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = x \cdot \frac{1}{2}(1-x)^{-\frac{1}{2}} \cdot (-1) + (1-x)^{\frac{1}{2}} \cdot 1$

$$= -\frac{x}{2\sqrt{1-x}} + \sqrt{1-x} = \frac{-x+2(1-x)}{2\sqrt{1-x}} = \frac{-3x+2}{2\sqrt{1-x}}$$

(j) (Chain Rule) $y = u^{-1}, u = (3x^2-5x+7) \Rightarrow \frac{dy}{dx} = -u^{-2} \cdot (6x-5) = -\frac{6x-5}{(3x^2-5x+7)^2}$

(k) (Chain Rule) $y = u^{\frac{1}{3}}, u = 2x+5 \Rightarrow \frac{dy}{dx} = \frac{1}{3}u^{-\frac{2}{3}} \cdot 2 = \frac{2}{3\sqrt[3]{(2x+5)^2}}$

(l) (Product + chain rule) $\frac{dy}{dx} = (2x-1)^3 \cdot 4x^3 + (x^4+1) \cdot 3 \cdot (2x-1)^2 \cdot 2$

$$= 2(2x-1)^2\left[2x^3(2x-1) + 3(x^4+1)\right]$$
$$= 2(2x-1)^2\left[7x^4 - 2x^3 + 3\right]$$

(m) (Chain Rule) $y = u^{\frac{1}{2}}, u = 3x^2-2 \Rightarrow \frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \cdot 6x = \frac{3x}{\sqrt{3x^2-2}}$

(n) (Quotient rule) $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x\cdot(x+2)-1\cdot x^2}{(x+2)^2}=\frac{x^2+4x}{(x+2)^2}$

(o) (Quotient rule) $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1\cdot(x-1)-1\cdot(x+1)}{(x-1)^2}=-\frac{2}{(x-1)^2}$

2. **(a)** $y=x^{-4}\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=-4\cdot x^{-5}=-\frac{4}{x^5}$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x}=-4\cdot x^{-5}\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-4\cdot-5\cdot x^{-6}=\frac{20}{x^6}$

3. **(a)** $(x-1)(4x^2+4x+1)=4x^3+4x^2+x-4x^2-4x-1=4x^3-3x-1$

$\frac{\mathrm{d}y}{\mathrm{d}x}=12x^2-3$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x}=(x-1)\cdot 2(2x+1)\cdot 2+(2x+1)^2\cdot 1=(x-1)(8x+4)+(2x+1)^2$

$=8x^2-4x-4+4x^2+4x+1=12x^2-3$

4. **(a)** Using the quotient rule:

$$f'(x)=\frac{(2x-3)\cdot(x+1)^2-2(x+1)(x^2-3x+4)}{(x+1)^4}$$
$$=\frac{(2x-3)\cdot(x+1)-2(x^2-3x+4)}{(x+1)^3}$$
$$=\frac{2x^2-x-3-2x^2+6x-8}{(x+1)^3}=\frac{5x-11}{(x+1)^3}$$

(b) Using the quotient rule again:

$$f''(x)=\frac{5\cdot(x+1)^3-3(x+1)^2\cdot(5x-11)}{(x+1)^6}$$
$$=\frac{5(x+1)-3(5x-11)}{(x+1)^4}$$
$$=\frac{5x+5-15x+33}{(x+1)^4}=\frac{-10x+38}{(x+1)^4}$$

5. $f'(x) = \dfrac{1 \cdot (x+a) - 1 \cdot (x-a)}{(x+a)^2}$

$= \dfrac{2a}{(x+a)^2}$

$f''(x) = \dfrac{0 \cdot (x+a)^2 - 2a \cdot 2 \cdot (x+a)}{(x+a)^4}$

$= -\dfrac{4a}{(x+a)^3}$

6. Using the product rule:

$y = x \cdot (x+1)^{\frac{1}{2}} \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = x \cdot \dfrac{1}{2}(x+1)^{-\frac{1}{2}} + (x+1)^{\frac{1}{2}} \cdot 1 =$

$= \dfrac{x}{2\sqrt{x+1}} + \sqrt{x+1} = \dfrac{x + 2(x+1)}{2\sqrt{x+1}} = \dfrac{3x+2}{2\sqrt{x+1}}$

7. We can see that $y = x^2(x^2 - 6)$ and the graphs of this function is negative for $-\sqrt{6} < x < \sqrt{6}$, and therefore it is negative for $0 < x < 1$, which is contained in the previous one.

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 12x = 4x(x^2 - 3)$ is negative for $0 < x < \sqrt{3}$

and therefore in the interval requested.

$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 - 12 = 12(x^2 - 1)$ which is also negative for $0 < x < 1$.

However, $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} = 24x$ is positive for all positive x and thus for $0 < x < 1$.

8. **(a)** Using Pythagoras' theorem we get:

$h = \sqrt{(b+1)^2 + b^2} = \sqrt{2b^2 + 2b + 1}$

(b) Using the chain rule:

$h = \sqrt{u} = u^{\frac{1}{2}}, u = 2b^2 + 2b + 1 \Rightarrow \dfrac{\mathrm{d}h}{\mathrm{d}b} = \dfrac{1}{2}u^{-\frac{1}{2}} \cdot (4b+2) = \dfrac{2b+1}{\sqrt{2b^2 + 2b + 1}}$