13.2.3 Letting $p = (1+abc)^{-1}$ we have $x = (1-b+ab)p$, $y = (1-c+bc)p$, $z = (1-a+ca)p$.

13.2.4 The three-equation system may be written in matrix form as

$$\begin{bmatrix} 1 & -1 & 0 \\ -c_1 & 1 & c_1 \\ -t_1 & 0 & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \\ T \end{bmatrix} = \begin{bmatrix} I+G \\ c_0 \\ t_0 \end{bmatrix}.$$

Solution by Cramer's rule gives the same answers as for Problem 2–1: see "Solutions to Problems".

13.3 Inner products

13.3.1 Since $\mathbf{b}^{\mathrm{T}}\mathbf{a} = \mathbf{a}^{\mathrm{T}}\mathbf{b}$, $(\lambda\mathbf{a}+\mu\mathbf{b})^{\mathrm{T}}(\lambda\mathbf{a}+\mu\mathbf{b}) = \lambda^2\mathbf{a}^{\mathrm{T}}\mathbf{a} + 2\lambda\mu\mathbf{a}^{\mathrm{T}}\mathbf{b} + \mu^2\mathbf{b}^{\mathrm{T}}\mathbf{b}$. **L3** now follows from the fact that $\mathbf{x}^{\mathrm{T}}\mathbf{x} = \|\mathbf{x}\|^2$ for every vector $\mathbf{x}$.

13.3.2 $|\mathbf{a}^{\mathrm{T}}\mathbf{b}| \le 1$ by **L4**, so $-1 \le \mathbf{a}^{\mathrm{T}}\mathbf{b} \le 1$. Examples:

$$\text{(a)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \text{(b)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}; \quad \text{(c)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \text{(d)} \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}, \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}.$$

13.3.3 Immediate from **L3** and the fact that $(-1)^2 = 1$.

13.3.4 For any invertible matrix $\mathbf{A}$, $\det(\mathbf{A}^{-1}) = (\det\mathbf{A})^{-1}$ and $\det(\mathbf{A}^{\mathrm{T}}) = \det\mathbf{A}$. In the special case where $\mathbf{A}^{-1} = \mathbf{A}^{\mathrm{T}}$, $(\det\mathbf{A})^{-1} = \det\mathbf{A}$, so $\det\mathbf{A} = \pm 1$.

Examples: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

13.3.5 Denoting the matrix $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ by $\mathbf{A}$, we see that

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

13.3.6 For example $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, which is not an orthogonal matrix because it is not square.

13.3.7 $\lambda = \frac{1}{\sqrt{2}}$, $\mu = \frac{1}{3}$, $\nu = \frac{1}{3\sqrt{2}}$ or their negatives.

13.4 Quadratic forms and symmetric matrices

13.4.1 If $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ then $\mathbf{a}\mathbf{a}^{\mathrm{T}} = \begin{bmatrix} a_1^2 & a_1a_2 \\ a_1a_2 & a_2^2 \end{bmatrix}$.

If $\mathbf{a}$ is an n–vector, $\mathbf{a}\mathbf{a}^{\mathrm{T}}$ is a symmetric $n \times n$ matrix.

13.4.2

$$\begin{bmatrix} \sum_{i=1}^{n} x_{1i}^2 & \sum_{i=1}^{n} x_{1i}x_{2i} \\ \sum_{i=1}^{n} x_{1i}x_{2i} & \sum_{i=1}^{n} x_{2i}^2 \end{bmatrix}$$

13.4.3 Let $\mathbf{C} = \mathbf{B}^{\mathrm{T}}\mathbf{A}\mathbf{B}$. Then $\mathbf{C}^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}(\mathbf{B}^{\mathrm{T}})^{\mathrm{T}}$. $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$ by assumption and $(\mathbf{B}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{B}$ always, so $\mathbf{C}^{\mathrm{T}} = \mathbf{C}$ as required.

13.4.4 $q(x_1, x_2, x_3) = x_1^2 + (x_2 - \frac{1}{2}x_3)^2 + \frac{3}{4}x_3^2 \geq 0$. If $q(x_1, x_2, x_3) = 0$ then x_1, $x_2 - \frac{1}{2}x_3$ and x_3 are all 0, so x_2 is also 0. Hence q is positive definite.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

13.4.5 The matrix $\mathbf{A}$ is $\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$, which has positive diagonal entries and determinant 2.

13.4.6 (a) $t > \sqrt{2}$, (b) $t = \sqrt{2}$, (c) $t < -\sqrt{2}$, (d) $t = -\sqrt{2}$, (e) $-\sqrt{2} < t < \sqrt{2}$.

13.4.7 Positive definite, indefinite, negative semidefinite.

13.4.8 The fact that $\mathbf{C}^{\mathrm{T}}\mathbf{C}$ is symmetric follows immediately from the rules $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ and $(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A}$. For any k–vector $\mathbf{w}$, $\mathbf{w}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{C}\mathbf{w} = \|\mathbf{C}\mathbf{w}\|^2 \geq 0$. In particular, $\mathbf{w}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{C}\mathbf{w} > 0$ if $\mathbf{C}\mathbf{w} \neq \mathbf{0}$, which happens if $\mathbf{w} \neq \mathbf{0}$ and the columns of $\mathbf{C}$ are linearly independent. Thus $\mathbf{C}^{\mathrm{T}}\mathbf{C}$ is always positive semidefinite, and is positive definite if the columns of $\mathbf{C}$ are linearly independent, which requires that $k \leq n$.

14 FUNCTIONS OF SEVERAL VARIABLES

14.1 Partial derivatives

14.1.1 (a) $\begin{bmatrix} 3 \\ 12y^2 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 24y \end{bmatrix}$.

(b) $\begin{bmatrix} 3x^2 \ln y + 12xy^3 + 2e^{2x}y \\ x^3/y + 18x^2y^2 + e^{2x} \end{bmatrix}$,

$$\begin{bmatrix} 6x \ln y + 12y^3 + 4e^{2x}y & 3x^2/y + 36xy^2 + 2e^{2x} \\ 3x^2/y + 36xy^2 + 2e^{2x} & -x^3/y^2 + 36x^2y \end{bmatrix}.$$

(c) $-(x^2 + 4y^2)^{-3/2} \begin{bmatrix} x \\ 4y \end{bmatrix}$, $2(x^2 + 4y^2)^{-5/2} \begin{bmatrix} x^2 - 2y^2 & 6xy \\ 6xy & -2x^2 + 16y^2 \end{bmatrix}$.

(d) $\begin{bmatrix} (1 - 2x - 8y)e^{-2x} + e^{3y} \\ 4e^{-2x} + (4 - 3x - 12y)e^{-3y} \end{bmatrix}$,

$$\begin{bmatrix} 4(-1 + x + 4y)e^{-2x} & -8e^{-2x} - 3e^{-3y} \\ -8e^{-2x} - 3e^{-3y} & 3(-8 + 3x + 12y)e^{-3y} \end{bmatrix}.$$

14.1.2 (a) $\begin{bmatrix} 6x \\ 10y^4 \end{bmatrix}$, $\begin{bmatrix} 6 & 0 \\ 0 & 40y^3 \end{bmatrix}$; $\begin{bmatrix} 6 \\ 160 \end{bmatrix}$, $\begin{bmatrix} 6 & 0 \\ 0 & -320 \end{bmatrix}$.

(b) $\begin{bmatrix} 6xy^3 + 6x^2y^2 \\ 9x^2y^2 + 4x^3y \end{bmatrix}$,

$$\begin{bmatrix} 6y^3 + 12xy^2 & 18xy^2 + 12x^2y \\ 18xy^2 + 12x^2y & 18x^2y + 4x^3 \end{bmatrix}; \quad \begin{bmatrix} -24 \\ 28 \end{bmatrix}, \quad \begin{bmatrix} 0 & 48 \\ 48 & -32 \end{bmatrix}.$$