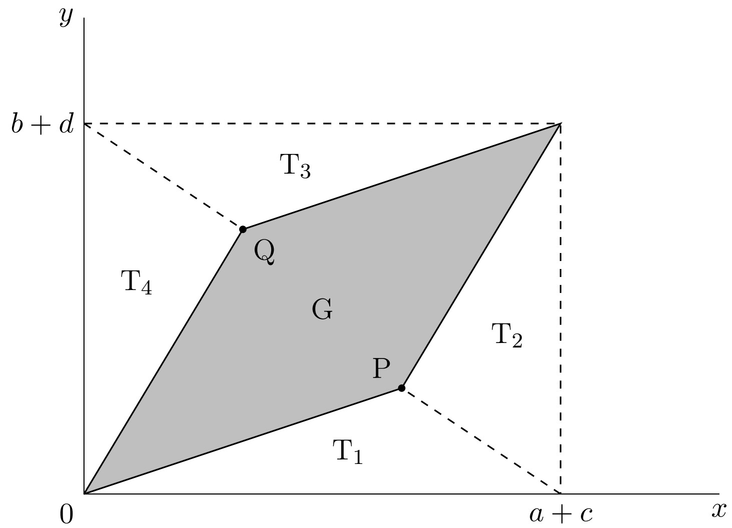
(iv) If $\mathbf{A}$ is singular then c_3 must be as in (iii). So if we replace the $(3,3)$ entry of $\mathbf{A}$ by any number other than c_3, then $\mathbf{A}$ becomes invertible.

12–4. From Problem 11–4, $\mathbf{Bx} = \mathbf{y}$, where $\mathbf{B} = \mathbf{I} - \mathbf{A}$. If there is to be a unique $\mathbf{x}$ for any $\mathbf{y}$, then $\mathbf{B}$ must be invertible and $\mathbf{x} = \mathbf{B}^{-1}\mathbf{y}$.

In addition, it is given that $\mathbf{y}$ has non-negative components. To ensure that $\mathbf{x}$ has non-negative components for every such $\mathbf{y}$ it is necessary that all entries of $\mathbf{B}^{-1}$ be non-negative. For suppose that $\mathbf{B}^{-1}$ had some negative entry, say the $(2,3)$ entry. By taking $\mathbf{y}$ to be the vector with third component 1 and zeros elsewhere, we see that the second component of $\mathbf{x}$ is negative.

13 DETERMINANTS AND QUADRATIC FORMS

13–1. Let P be the point (a,b), Q the point (c,d).



In the diagram, the area of each of the triangles T_1 and T_3 is $\frac{1}{2}(a+c)b$ by the half-base-times-height formula. Similarly, the area of each of the triangles T_2 and T_4 is $\frac{1}{2}(b+d)c$. Hence the area of G is

$$\begin{aligned}(a+c)(b+d) - (a+c)b - (b+d)c &= (a+c)d - (b+d)c \\ &= ad - bc.\end{aligned}$$

If we exchange the positions of P and Q, the area of G becomes $cb - da$, which equals $-(ad - bc)$. Thus the general formula for the area of G is $|ad - bc|$.

13-2. (i) $\det \mathbf{A} = c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$.

Let $\mathbf{C}$ be the matrix obtained from $\mathbf{A}$ by replacing its $(3,3)$ entry by $c_3 + \delta$. Replacing c_3 by $c_3 + \delta$ in the expression just given for $\det \mathbf{A}$, we see that $\det \mathbf{C} = \det \mathbf{A} + \delta \det \mathbf{B}$. By our assumptions about $\mathbf{A}$ and $\mathbf{B}$, $\det \mathbf{A} = 0$ and $\det \mathbf{B} \neq 0$. Hence $\det \mathbf{C} \neq 0$ if $\delta \neq 0$.

(ii) Let $\mathbf{A}$ be a singular 2×2 matrix and let $\mathbf{C}$ be the matrix obtained by adding x to each of its diagonal entries. Since $\det \mathbf{A} = 0$, $\det \mathbf{C} = tx + x^2$, where t is the sum of the diagonal entries of $\mathbf{A}$. If $t = 0$, $\det \mathbf{C} > 0$ for any non-zero x; if $t \neq 0$, $\det \mathbf{C} > 0$ whenever x has the same sign as t; in each case, $|x|$ can be as small as we please.

Now suppose we have a singular 3×3 matrix $\mathbf{A}$. As in (i), we denote the 2×2 leading principal submatrix of $\mathbf{A}$ by $\mathbf{B}$. If $\mathbf{B}$ is invertible then, as in (i), we can make $\mathbf{A}$ invertible by an arbitrarily small change to its $(3,3)$ entry. If $\mathbf{B}$ is singular we can apply the proposition in the 2×2 case, making $\mathbf{B}$ invertible by arbitrarily small changes to its diagonal entries; we can then use (i) as before. This proves the proposition for 3×3 matrices.

For the 4×4 case, if necessary we apply the proposition for the 3×3 case to ensure that the leading principal submatrix of order 3 is nonsingular. Then, by a similar argument to (i), the 4×4 matrix can be made invertible by an arbitrarily small change to its $(4,4)$ entry. In the same way, the proposition for the 4×4 case can then be used to prove it for the 5×5 case, and so on.

(iii) It is easy to see from the expansion formulae that small changes in the entries of a matrix cause only small changes in the determinant. Therefore, arbitrarily small changes in diagonal entries are not enough to transform a matrix with nonzero determinant into a singular matrix.

13–3. The cost of producing each unit of gross output of good j is

$$c_j + p_1 a_{1j} + p_2 a_{2j} + \ldots + p_n a_{nj}.$$

If all industries exactly break even, then this expression must be equal to p_j for all j. Hence we may write the break-even condition for all industries as the single vector equation $\mathbf{c}+\mathbf{A}^{\mathrm{T}}\mathbf{p} = \mathbf{p}$, or $(\mathbf{I} - \mathbf{A}^{\mathrm{T}})\mathbf{p} = \mathbf{c}$.

Now observe that $\mathbf{I} - \mathbf{A}^{\mathrm{T}} = (\mathbf{I} - \mathbf{A})^{\mathrm{T}}$. Denoting $\mathbf{I} - \mathbf{A}$ by $\mathbf{B}$ as in Problems 11–4 and 12–4, we may write the break-even condition as $\mathbf{B}^{\mathrm{T}}\mathbf{p} = \mathbf{c}$. If there is to be a unique $\mathbf{p}$ for any $\mathbf{c}$, then $\mathbf{B}^{\mathrm{T}}$ must be invertible and $\mathbf{p} = (\mathbf{B}^{\mathrm{T}})^{-1}\mathbf{c}$. In addition, it is given that $\mathbf{c}$ has non-negative components. To ensure that $\mathbf{p}$ has non-negative components for every such $\mathbf{c}$, it is necessary that $(\mathbf{B}^{\mathrm{T}})^{-1}$ has non-negative entries. This follows from an argument similar to that given in Problem 12–4.

Finally, observe that $\mathbf{B}^{\mathrm{T}}$ is invertible if and only if $\mathbf{B}$ is invertible. If $\mathbf{B}$ is invertible then $(\mathbf{B}^{\mathrm{T}})^{-1} = (\mathbf{B}^{-1})^{\mathrm{T}}$; in particular, all entries of $(\mathbf{B}^{\mathrm{T}})^{-1}$ are non-negative if and only if all entries of $\mathbf{B}^{-1}$ are non-negative. Thus $\mathbf{A}$ has the properties required here if and only if it has the properties required in Problem 12–4.

13–4. (i) The ith component of $\mathbf{y} - \mathbf{X}\mathbf{b}$ is $y_i - b_1 x_{1i} - b_2 x_{2i}$. The result follows.

(ii) $\mathbf{y} - \mathbf{X}\mathbf{b} = \mathbf{y} - \mathbf{X}\mathbf{b}^* + \mathbf{X}(\mathbf{b}^* - \mathbf{b}) = \mathbf{p} + \mathbf{q}$ where $\mathbf{p} = \mathbf{X}(\mathbf{b}^* - \mathbf{b})$, $\mathbf{q} = \mathbf{y} - \mathbf{X}\mathbf{b}^*$ and $\mathbf{p}^{\mathrm{T}}\mathbf{q} = 0$. The result then follows from that of Exercise 13.3.1.

(iii) $(*)$ can be written as $\left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)\mathbf{b}^* = \mathbf{X}^{\mathrm{T}}\mathbf{y}$. Since $\mathbf{X}^{\mathrm{T}}\mathbf{X}$ is invertible, there is only one vector $\mathbf{b}^*$ which satisfies $(*)$; this is given by $\mathbf{b}^* = \left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{y}$.

(iv) The answer to (ii) expresses $Q(\mathbf{b})$ as the sum of two terms, only the second of which depends on $\mathbf{b}$. Since $\mathbf{X}^{\mathrm{T}}\mathbf{X}$ is positive definite, this second term is positive if $\mathbf{b} \neq \mathbf{b}^*$, zero if $\mathbf{b} = \mathbf{b}^*$. Hence $Q(\mathbf{b})$ is minimised when $\mathbf{b} = \mathbf{b}^*$.

14 FUNCTIONS OF SEVERAL VARIABLES

14–1. (i) $\partial z/\partial x = y$ and $\partial z/\partial y = x$, so the equation of the tangent plane is

$$z = 12 + 3(x - 4) + 4(y - 3).$$

When $x = 4 + h$ and $y = 3 + k$, the value of z given by the tangent plane is $12 + 3h + 4k$.